

HIGHER ORDER CONVERGENCE RATES IN THEORY OF HOMOGENIZATION I: EQUATIONS OF NON-DIVERGENCE FORM

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ABSTRACT. We establish higher order convergence rates in the theory of periodic homogenization of both linear and fully nonlinear uniformly elliptic equations of non-divergence form. The rates are achieved by involving higher order correctors which fix the errors occurring both in the interior and on the boundary layer of our physical domain. The proof is based on a viscosity method and a new regularity theory which captures the stability of the correctors with respect to the shape of our limit profile.

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1. INTRODUCTION

We establish higher order convergence rates in the theory of periodic homogenization of both linear and fully nonlinear uniformly elliptic equations of non-divergence form. It is known that the equations containing highly oscillating

variables $\frac{x}{\varepsilon}$, where the oscillation takes place periodically in the microscopic scale, exhibit a limiting behavior as $\varepsilon \rightarrow 0$. More precisely, for the following ε -problems with linear operators,

$$(L_\varepsilon) \quad \begin{cases} a_{ij}\left(\frac{x}{\varepsilon}\right) D_{ij} u^\varepsilon = f & \text{in } \Omega, \\ u^\varepsilon = g & \text{on } \partial\Omega, \end{cases}$$

the solutions u^ε converge to a function u as $\varepsilon \rightarrow 0$, which solves a boundary value problem

$$(\bar{L}) \quad \begin{cases} \bar{a}_{ij} D_{ij} u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

whose operator is homogenous (i.e., the matrix (\bar{a}_{ij}) is constant) with respect to the environment. For more details, one may refer to [BLP] and [JKO]. A similar behavior does exist also when the operator consists of nonlinearity, namely,

$$(F_\varepsilon) \quad \begin{cases} F(D^2 u^\varepsilon, x, \frac{x}{\varepsilon}) = 0 & \text{in } \Omega, \\ u^\varepsilon = g & \text{on } \partial\Omega. \end{cases}$$

As in the linear case, the solutions u^ε exhibit a limiting behavior, and the limit profile u turns out to be a solution of the following PDE,

$$(\bar{F}) \quad \begin{cases} \bar{F}(D^2 u, x) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

where \bar{F} is no longer oscillatory in the microscopic scale. For more details, see [E2].

In this paper, we give a quantitative analysis on the rate of convergence between the solution u^ε and its limit profile u , and we further accelerate the rate by involving appropriate corrector functions for both interior and boundary layer of the physical domain. Finally we end up with a rigorous justification of the following two scale expansion of the solution u^ε :

$$(1.0.1) \quad u^\varepsilon(x) = u(x) + \varepsilon(w_1^\varepsilon(x) + z_1^\varepsilon(x)) + \cdots + \varepsilon^m(w_m^\varepsilon(x) + z_m^\varepsilon(x)) + O(\varepsilon^{m-1}),$$

where w_k^ε and z_k^ε respectively are the k -th order correctors which fix the error occurring in the interior and on the boundary layer respectively, and m is the positive integer related to the regularity of the operator of the ε -problem. The above expression is explicit if the ε -problem is linear, but rather implicit when a nonlinearity comes in.

The study of higher order convergence rate in homogenization theory is new, to our best knowledge, for second order uniformly elliptic equations in non-divergence form. To obtain higher order convergence rates for the linear equations of divergence type, the expansion of the fundamental solution with respect to ε -parameter plays a crucial role, since the solution to the ε -problem admits an integral representation with the fundamental solution. On the contrary, the (nonlinear) equations of non-divergence type operators do not have any integral representation of the solution. In this paper, we develop viscosity method for the higher order approximation based on comparison principles and regularity theories, which can be found in [CC] and [CIL].

1.1. Linear equations. Set Ω to be a bounded domain in \mathbb{R}^n with $C^{m+2,\alpha}$ boundary and let $f \in C^{m,\alpha}(\overline{\Omega})$ and $g \in C^{m+2,\alpha}(\overline{\Omega})$ for some exponent $0 < \alpha \leq 1$ and an integer $m \geq 2$. We suppose that $A(y) = (a_{ij}(y))$, $1 \leq i, j \leq n$ is a symmetric matrix-valued function defined in \mathbb{R}^n satisfying the following hypotheses:

(L1) (Periodicity) A is 1-periodic; i.e.,

$$A(y + k) = A(y) \quad (y \in \mathbb{R}^n, k \in \mathbb{Z}^n).$$

(L2) (Uniform Ellipticity) There are some $0 < \lambda \leq \Lambda$ such that

$$\lambda |\xi|^2 \leq a_{ij}(y) \xi_i \xi_j \leq \Lambda |\xi|^2 \quad (\xi \in \mathbb{R}^n, y \in \mathbb{R}^n).$$

(L3) (Regularity) There is a constant $\sigma > 0$ such that

$$\|A\|_{C^{m,\alpha}(\mathbb{R}^n)} \leq \sigma.$$

Our main result for linear equations can be summarized as follows.

Theorem 1.1.1 (Main Theorem I). *Let $m \geq 2$ be an integer and suppose that (L_ε) satisfies the structure conditions (L1)-(L3). Assume that $\{u^\varepsilon\}_{\varepsilon>0}$ is the family of the solutions of (L_ε) and u is the homogenized limit of $\{u^\varepsilon\}_{\varepsilon>0}$ which solves (\bar{L}) . Then there are interior correctors w_k^ε and boundary layer correctors z_k^ε for $k = 1, \dots, m$ such that*

$$(1.1.1) \quad \|u^\varepsilon - \eta_m^\varepsilon - \theta_m^\varepsilon\|_{L^\infty(\Omega)} \leq C\varepsilon^{m-1}$$

for any $\varepsilon \in (0, 1)$, where

$$\eta_m^\varepsilon = u + \varepsilon w_1^\varepsilon + \varepsilon^2 w_2^\varepsilon + \dots + \varepsilon^m w_m^\varepsilon, \quad \theta_m^\varepsilon = \varepsilon z_1^\varepsilon + \varepsilon^2 z_2^\varepsilon + \dots + \varepsilon^m z_m^\varepsilon$$

on $\overline{\Omega}$ and C is a positive constant depending only on $n, m, \alpha, \sigma, \lambda, \Lambda, \Omega, \|f\|_{C^{m,\alpha}(\overline{\Omega})}$ and $\|g\|_{C^{m+2,\alpha}(\overline{\Omega})}$.

1.2. Fully nonlinear equations. Set Ω to be a bounded domain of \mathbb{R}^n with $\partial\Omega \in C^{m+2,1}$ and let $g \in C^{m+2,1}(\overline{\Omega})$. Suppose that $F \in C^m(\mathcal{S}^n \times \overline{\Omega} \times \mathbb{R}^n)$ possesses the following properties.

(F1) (Periodicity) F is 1-periodic in y -variable; i.e.,

$$F(M, x, y + k) = F(M, x, y) \quad ((M, x, y) \in \mathcal{S}^n \times \overline{\Omega} \times \mathbb{R}^n, k \in \mathbb{Z}^n).$$

(F2) (Uniform Ellipticity) There are some constants $0 < \lambda \leq \Lambda$ such that for any $M, N \in \mathcal{S}^n$ with $N \geq 0$,

$$\lambda \|N\| \leq F(M + N, x, y) - F(M, x, y) \leq \Lambda \|N\| \quad (x \in \overline{\Omega}, y \in \mathbb{R}^n).$$

(F3) (Regularity) There is a constant $\sigma > 0$ and a nondecreasing function $\tau : (0, \infty) \rightarrow [1, \infty)$ such that for any $L > 0$,

$$\|F\|_{C^{m,1}(\bar{B}_L \times \overline{\Omega} \times \mathbb{R}^n)} \leq \tau_L,$$

where for any $M \in \mathcal{S}^n$,

$$\tau_{\|M\|} \leq \sigma(1 + \|M\|).$$

(F4) (Concavity) F is concave in M -variable.

Before we state our main result in the framework of fully nonlinear equations, we would like to make a remark about the above hypotheses. The periodicity, uniform ellipticity and regularity assumptions, (F1), (F2) and (F3) respectively, are essential and correspond to (L1), (L2) and (L3) respectively for linear equations. There are two reasons to impose the hypothesis (F4). First, it is a sufficient condition when combined with (F3) to get an interior $C^{2,\alpha}$ regular solution for $F(D^2v, y) = 0$ (see B.0.2 (e)), by which we obtain a $C^{2,\alpha}$ interior corrector of each order. Second, it yields the homogenised operator \bar{F} to be concave as well, from which we get $C^{2,\alpha}$ regularity of the limit profile u , and thus we are able to boost its regularity up to $C^{m+2,\alpha}$ at the end, which is necessary to proceed with our argument. To obtain the former, we are safe under a weaker assumption that F admits an interior $C^{2,\alpha}$ estimate (see Chapter 8 of [CC] for the exact definition of this terminology). However, it is yet unclear whether or not the effective operator \bar{F} would also inherit this property; it would be an interesting question.

Our main result for fully nonlinear equations is summarized in the following theorem.

Theorem 1.2.1 (Main Theorem II). *Let $m \geq 2$ and assume that F satisfies the structure conditions (F1)–(F4), $g \in C^{m+2,1}(\bar{\Omega})$ and $\partial\Omega \in C^{m+2,1}$. Then there are interior correctors w_k^ε for $k = 1, \dots, [\frac{m}{2}] + 1$ and the boundary layer corrector θ_m^ε such that for any $\varepsilon_* \in (0, 1)$,*

$$(1.2.1) \quad \|u^\varepsilon - \eta_m^\varepsilon - \theta_m^\varepsilon\|_{L^\infty(\Omega)} \leq C\varepsilon^{[\frac{m}{2}]}, \quad \forall \varepsilon \in (0, \varepsilon_*],$$

where

$$\eta_m^\varepsilon = u + \varepsilon w_1^\varepsilon + \varepsilon^2 w_2^\varepsilon + \dots + \varepsilon^{[\frac{m}{2}]+1} w_{[\frac{m}{2}]+1}^\varepsilon$$

on $\bar{\Omega}$ and $C > 0$ depends only on $n, m, \varepsilon_*, \sigma, \lambda, \Lambda, F, g$ and Ω .

1.3. Main steps. In this subsection, we summarize the main strategies of this paper and make a few remarks on the key features observed in achieving the rates.

Higher order correctors and regularity theory In order to find the next order approximation, we consider the linearized operator near the previous approximation, which turns out to be still in the same class of the previous one. For this reason, we are able to employ the basic method for the existence and the regularity of the correctors for each order in an inductive manner. The relationship between the current approximation and the next one is rather clear in the framework of linear equations. However, it is very complicated in nonlinear settings. We have overcome this difficulty by carrying out an interesting regularity theory, which captures the stability of correctors with respect to the shape of the limit profile, but not to the physical variable x . It will be a new result in the theory of regularity for fully nonlinear equations.

Induction arguments and compatibility conditions We point out the key feature when carrying out our induction process that it consists of two sub-steps at each main step. To be precise, let us suppose that we are in k -th main step. Then we first improve the approximation, say $w_{k-1}^\varepsilon(x) + z_{k-1}^\varepsilon(x)$, made in $(k-1)$ -th step by constructing the k -th order globally periodic corrector, $\eta_k^\varepsilon(x)$, at the interior and bending the correctors based on the shape of the limit profile. The improved interior approximation, i.e., $w_k^\varepsilon(x)$, then creates new errors of order $O(\varepsilon^k)$ away from the given boundary data. It leads us to the second sub-step which involves

the construction of the k -th order boundary layer corrector $z_k^\varepsilon(x)$, by which we fix the new error occurring on the boundary.

Additionally it is noteworthy to observe that at each step of finding the k -th order interior corrector w_k^ε we encounter a compatibility condition which determines uniquely the $(k-2)$ -th order interior corrector. To illustrate this fact, we consider the basic cell problem

$$F(D_y^2 w(y; M) + M, y) = \mu(M) \quad \text{in } \mathbb{R}^n$$

(for linear equations, $F(D_y^2 w(y; M) + M, y) = a_{ij}(y)D_{y_i y_j} w(y; M) + a_{ij}(y)M_{ij}$). In order to get $\mu = 0$ on the right hand side, we need to choose a very specific quadratic data, namely $M = D_x^2 u(x)$, which is the Hessian of the solution of the effective equation. This kind of relationship continuously appears at each k -th step, and the compatibility condition turns out to be the solvability of a boundary value problem whose operator is nothing but the effective operator. Moreover, it illustrates the reason why the higher order asymptotic expansion (1.0.1) starts from ε -order with a non-trivial function $w_1^\varepsilon(x) + z_1^\varepsilon(x)$, which would be rather unclear if we restrict ourselves to perturbed test function method (see [E1, E2]). Finally, we point out that the compatibility issue is crucial for achieving higher order convergence rate for equations with divergence type operators as well, and it is the same in this situation that the compatibility condition of k -th order corrector uniquely determines the $(k-2)$ -th order corrector. It seems to be related to the invariance of quadratic rescaling of the ε -problem, and we will discuss this phenomenon in more detail in our forthcoming paper.

Linearization and coupling effects Here we address the two main differences between the linear and fully nonlinear settings. First we see that the asymptotic expansion (1.0.1) is made inside of the operator for fully nonlinear case rather than “outside” as in the case of linear equations. This creates an additional error as we perturb our fully nonlinearity, and forces us to take this error into account when constructing the equations for the interior correctors; the rightmost term of Φ_k in (3.3.8) is exactly the term which represents this error, and one may notice that there is no such a term in the linear case (2.2.11). In spite of this difficulty, a more sophisticated analysis shows that it can be overcome and has no influence on the determination of order of the convergence rate.

What turns out to be severe is the coupling effect of the fast variable $y = \varepsilon^{-1}x$ and the slow variable x of the interior correctors $w_k(y, x)$ in the fully nonlinear case. This coupling effect becomes more apparent if we see through the linear case first. In the linear case, the interior correctors w_k can be represented in the form of (2.2.10), which is the summation of the functions whose (y, x) -variable is decoupled, in other words, separated. Hence, it makes no difference between the regularity of $y \mapsto w_k(y, x)$ and $y \mapsto D_x^i w_k(y, x)$ for any $i \geq 1$. For this reason, we are allowed to construct the interior correctors as many as the regularity of the given data (i.e., a_{ij} , f , g and $\partial\Omega$ in (L_ε)), which in this case is m . As long as m correctors are involved, we are able to achieve $O(\varepsilon^{m-1})$ -rate in the linear case (see (1.1.1)).

On the contrast, the interior correctors $w_k(y, x)$ for fully nonlinear equations does not admit such a decoupled representation as (2.2.10). As a result, the function $y \mapsto D_x^i w_k(y, x)$ turns out to have a lower regularity than that of $y \mapsto w_k(y, x)$. The effect remains in the equation of the next order interior corrector $w_{k+1}(y, x)$, and make the regularity of $x \mapsto w_{k+1}(y, x)$ to decrease in “two steps” from that

of $x \mapsto w_k(y, x)$, while the decrement takes place only in “one step” in the linear case; note that $w_k(y, \cdot) \in C^{m-2k+2,1}(\overline{\Omega})$ for the fully nonlinear case (Lemma 3.3.2 (ii)) whereas $w_k(y, \cdot) \in C^{m-k+2,\alpha}(\overline{\Omega})$ (see the comment below (2.2.10)). This accounts for the reason why in the fully nonlinear case we could only obtain $\lfloor \frac{m}{2} \rfloor + 1$, which is a half of m , interior correctors, and thus, end up with $O(\varepsilon^{\lfloor \frac{m}{2} \rfloor})$ -rate.

As a final remark, we point out that as long as r correctors are found, we are able to get $O(\varepsilon^{r-1})$ -rate in any cases.

1.4. Historical Background. Here we discuss the development of the homogenization theory of nonlinear first- and second-order partial differential equations in the periodic environments, and several results on the rate of convergence. For linear equations, we recommend the readers to refer to the book [BLP] by Bensoussan, Lions and Papanicolaou, and the references therein.

Lions, Papanicolaou and Varadhan [LPV] established the result for homogenization for periodic first order nonlinear (Hamilton-Jacobi) equations. The problem was revisited by Evans [E1, E2] who introduced the notion of perturbed test function and considered also second order equations such as (F_ε) . Caffarelli [C] proposed a different approach for the homogenization of fully nonlinear uniformly elliptic equations. Time-dependent problems were studied by Majda and Souganidis [MS] and Evans and Gomes [EG]. Ishii [I] considered the homogenization of almost periodic Hamilton-Jacobi equations, while Lions and Souganidis [LS] analyzed second order degenerate elliptic equations.

Several results were known about rates of convergence in the theory of periodic homogenization. For linear equations, the $O(\varepsilon)$ rate was established and proved to be optimal (see [BLP]). For the first order fully nonlinear equations, Capuzzo Dolcetta and Ishii [CI] proved a Hölder ($O(\varepsilon^\alpha)$) rate. For the second order fully nonlinear equations, Camilli and Marchi [CM] obtained a Hölder rate under the assumption that F is uniformly elliptic and convex. More recently, Caffarelli and Souganidis [CS] improved this result even for nonconvex nonlinearity by introducing the notion of δ -viscosity solutions, which plays a role of $C^{2,\alpha}$ -approximation of the corrector. As far as we know, there has been, however, no literature concerning higher order convergence rates for homogenization of both linear and nonlinear elliptic equations.

1.5. Outline. This paper is organized as follows. Section 2 is devoted to linear equations. We review the basic homogenization scheme via the viscosity method in Subsection 2.1. Interior and boundary layer correctors of higher order are obtained in Subsection 2.2. We present the proof of Main Theorem I in Subsection 2.3. Section 3 is devoted to fully nonlinear equations. The basic homogenization scheme of fully nonlinear equations is shown in Subsection 3.1. In Subsection 3.2 we investigate the regularity of the effective operator and the corrector function in the slow variable. In Subsection 3.3 we seek the higher order interior and boundary layer correctors, and finally prove Main Theorem II in Subsection 3.4.

1.6. Basic notations and terminologies.

- \mathcal{S}^n is the space of all $n \times n$ symmetric matrices. $\|M\|$ denotes the (L^2, L^2) -norm of M (i.e., $\|M\| = \sup_{|x|=1} |Mx|$). By $M \geq 0$ we indicate that all the eigenvalues of M is nonnegative.

- $B_r(x) = \{y : \|y - x\| < r\}$; x may be a point in \mathbb{R}^n or \mathcal{S}^n . Also, we abbreviate $B_r(0)$ by B_r .
- $Q_r(x) = (-r/2, r/2)^n \subset \mathbb{R}^n$. As above, by Q_r we denote $Q_r(0)$.
- A function φ on \mathbb{R}^n is said to be 1-periodic if $\varphi(y + k) = \varphi(y)$ for any $y \in \mathbb{R}^n$ and $k \in \mathbb{Z}^n$. Alternatively, we also write $\varphi \in \mathbb{T}^n$, where $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ is the n -dimensional torus.
- Given $\varphi, \psi \in C(\Omega)$, φ is said to touch ψ by above (resp., by below) at x_0 in Ω if $\varphi(x) \geq \psi(x)$ [resp., $\varphi(x) \leq \psi(x)$] for all $x \in \Omega$ and $\varphi(x_0) = \psi(x_0)$.
- $C^{k,\alpha}(\Omega)$ and $C^{k,\alpha}(\overline{\Omega})$ denote Hölder ($0 < \alpha < 1$) and Lipschitz ($\alpha = 1$) spaces. $\|\cdot\|_{C^{k,\alpha}(\Omega)}^*$ and $\|\cdot\|_{C^{k,\alpha}(\overline{\Omega})}^{(j)}$ are adimensional norms whose definition can be found in Chapter 4 of [GT].
- For a differentiable functional F on $\mathcal{S}^n \times \Omega$, by $D_p F$ [resp., $D_x F$] is the partial derivative in M -variable [resp., in x -variable].
- We use the summation convention of repeated indices.
- Unless otherwise stated, we always follow the following convention of constants: By c_n, C_n we denote dimensional constants. By c_0, c, C_0, C we denote the positive constants which depends only on the constants appearing in the structure conditions (L1)-(L3) or (F1)-(F4); i.e., $n, \alpha, \sigma, \lambda, \Lambda$. By C_{f_1, \dots, f_k} and $C(f_1, \dots, f_k)$ we denote positive constants depending on the constants in the structure conditions and further on f_1, \dots, f_k where f_i can be either a constant or a function.

2. LINEAR EQUATIONS IN NON-DIVERGENCE FORM

2.1. Basic homogenization scheme. Let us fix $\varepsilon > 0$. The coefficient matrix $(a_{ij}(\cdot/\varepsilon))$ of (L_ε) is uniformly elliptic in $\overline{\Omega}$ with constants λ and Λ , and belongs to $C^{m,\alpha}(\overline{\Omega})$. According to Proposition A.0.1 (f) and (e), there exists a unique solution $u^\varepsilon \in C^{m+2,\alpha}(\overline{\Omega})$ of (L_ε) .

Lemma 2.1.1. *Let $\{u^\varepsilon\}_{\varepsilon>0} \subset C^{m+2,\alpha}(\overline{\Omega})$ be the unique family which solve (L_ε) for each $\varepsilon > 0$. Then there is a function $u \in C(\overline{\Omega})$ and a subsequence $\{u^{\varepsilon_k}\}_{k=1}^\infty$ of $\{u^\varepsilon\}_{\varepsilon>0}$ such that $u^{\varepsilon_k} \rightarrow u$ uniformly in $\overline{\Omega}$ as $k \rightarrow \infty$.*

Proof. Since $(a_{ij}(\cdot/\varepsilon))$ satisfies (L2) with the constants λ and Λ , independent of the change of ε , we have $u^\varepsilon \in S(\lambda, \Lambda, f)$ in Ω for all $\varepsilon > 0$. Due to the assumption that $g \in C^{m,\alpha}(\overline{\Omega}) \subset C^\alpha(\overline{\Omega})$, g has the modulus of continuity ρ with $\rho(r) = [g]_{C^\alpha(\overline{\Omega})} r^\alpha$. Also by the assumption that $\partial\Omega \in C^{m+2,\alpha}$, Ω satisfies a uniform sphere condition (with radius $R > 0$). Thus, we are able to apply Theorem B.0.2 (d) to conclude that u^ε has a modulus of continuity ρ^* , which depends only on $n, \lambda, \Lambda, \|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \text{diam}(\Omega), R$ and ρ . As the modulus of continuity ρ^* is independent on ε , the family $\{u^\varepsilon\}_{\varepsilon>0}$ is equicontinuous on $\overline{\Omega}$.

Moreover, the a priori estimate (Proposition A.0.1 (a) with $b \equiv c \equiv 0$) of u^ε ascertains that the family of $\{u^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded by

$$\sup_{\varepsilon>0} \|u^\varepsilon\|_{L^\infty(\Omega)} \leq C \left(\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Omega)} \right),$$

where C depends only on λ, Λ and $\text{diam}(\Omega)$.

Now the conditions for the Arzela-Ascoli theorem are met, which ensures the existence of a subsequence $\{u^{\varepsilon_k}\}_{k=1}^\infty$ of $\{u^\varepsilon\}_{\varepsilon>0}$ which converges uniformly in $\overline{\Omega}$. \square

The function $u \in C(\overline{\Omega})$ will later turn out to be unique and satisfy (\bar{L}) in the classical sense. The next lemma plays a key role in proving this fact.

Lemma 2.1.2. *For each $M \in \mathcal{S}^n$ there exists a unique $\gamma \in \mathbb{R}^n$ for which the following equation admits a 1-periodic solution*

$$(2.1.1) \quad a_{ij}D_{y_i y_j} w + a_{ij}M_{ij} = \gamma \quad \text{in } \mathbb{R}^n.$$

Moreover, the solutions of (2.1.1) lie in $C^{2,\alpha}(\mathbb{R}^n)$ and are unique up to an additive constant.

To prove this lemma we consider the following penalized problem for $\delta \in (0, 1)$

Lemma 2.1.3. *Let $M \in \mathcal{S}^n$. There exists a unique bounded 1-periodic solution w^δ of*

$$(2.1.2) \quad a_{ij}D_{y_i y_j} w^\delta + a_{ij}M_{ij} - \delta w^\delta = 0 \quad \text{in } \mathbb{R}^n.$$

for each $\delta \in (0, 1)$. Moreover, w^δ lies in $C^{2,\alpha}(\mathbb{R}^n)$ with the estimate

$$(2.1.3) \quad \sup_{0 < \delta < 1} \|\delta w^\delta\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C \|M\|.$$

Proof. In view of Theorem B.0.1 (a) (with $F(N, p, r, y) = a_{ij}(y)N_{ij} + a_{ij}(y)M_{ij} - \delta r$), we know that (2.1.2) has a comparison principle. By the hypothesis (L2), all the eigenvalues of (a_{ij}) lie in the interval $[\lambda, \Lambda]$, which implies that

$$(2.1.4) \quad \|a_{ij}M_{ij}\|_{L^\infty(\mathbb{R}^n)} \leq n \left(\|A(y)\|_{C^\alpha(\mathbb{R}^n)} \right) \|M\| \leq n\sigma \|M\|.$$

It then follows that the constant functions $w_-^\delta = -n\sigma \|M\| / \delta$ and $w_+^\delta = n\sigma \|M\| / \delta$ are respectively a subsolution and a supersolution to (2.1.2) for each $\delta \in (0, 1)$. Thus, Perron's method (Theorem B.0.1 (b) with $F(N, p, v, y) = a_{ij}(y)N_{ij} + a_{ij}(y)M_{ij} - \delta v(y)$, $u_- = w_-^\delta$ and $u_+ = w_+^\delta$) ensures that there is a unique bounded 1-periodic viscosity solution $w^\delta \in C(\mathbb{R}^n)$. It is immediate that

$$(2.1.5) \quad \sup_{0 < \delta < 1} \|\delta w^\delta\|_{L^\infty(\mathbb{R}^n)} \leq n\sigma \|M\|.$$

Let us apply an interior Schauder estimate in a ball $B_1(y_0)$ for $y_0 \in \mathbb{R}^n$ (see Proposition A.0.1 (b) with $\Omega = B_1(y_0)$). Then $w^\delta \in C^{2,\alpha}(B_1(y_0))$ and there is c_0 such that

$$\|w^\delta\|_{C^{2,\alpha}(B_1(y_0))} \leq c_0 \left(\|w^\delta\|_{L^\infty(B_1(y_0))} + n\sigma \|M\| \right) \leq 2n\delta^{-1}c_0\sigma \|M\|.$$

Since y_0 was chosen in an arbitrary way, (2.1.3) is verified with $C = 2n\delta^{-1}c_0\sigma$. \square

We observe that the oscillation of w^δ is bounded independent of δ , although its L^∞ norm is not bounded in a uniform way.

Lemma 2.1.4. *Let $M \in \mathcal{S}^n$ and w^δ be the unique solution to (2.1.2). Then*

$$(2.1.6) \quad \sup_{0 < \delta < 1} \operatorname{osc}_{\mathbb{R}^n} w^\delta \leq C \|M\|.$$

Moreover,

$$(2.1.7) \quad \sup_{0 < \delta < 1} \|\tilde{w}^\delta\|_{C^{2,\alpha}(B_1(y_0))} \leq C \|M\|,$$

where $\tilde{w}^\delta := w^\delta - w^\delta(0)$.

Proof. Define

$$\hat{w}^\delta(y) := w^\delta(y) - \min_{\mathbb{R}^n} w^\delta \geq 0 \quad (y \in \mathbb{R}^n).$$

Note that the minimum of w^δ is achieved because w^δ is bounded and $C^{2,\alpha}$ in \mathbb{R}^n . Therefore, \hat{w}^δ is well-defined and indeed lies in $C^{2,\alpha}(\mathbb{R}^n)$. By the same reason, w^δ and \hat{w}^δ achieve their global maximums. Moreover, plugging \hat{w}^δ into (2.1.2) we obtain

$$(2.1.8) \quad a_{ij}D_{y_i y_j} \hat{w}^\delta - \delta \hat{w}^\delta = \delta \min_{\mathbb{R}^n} w^\delta - a_{ij}M_{ij} \quad \text{in } \mathbb{R}^n.$$

The following identity motivates us to consider \hat{w}^δ ;

$$\text{osc}_{\mathbb{R}^n} w^\delta = \sup_{\mathbb{R}^n} w^\delta - \inf_{\mathbb{R}^n} w^\delta = \sup_{\mathbb{R}^n} \hat{w}^\delta.$$

Let us restrict our domain to $B_{\sqrt{n}}(y_0)$ where y_0 is an arbitrary point in \mathbb{R}^n . Note that $B_{\sqrt{n}/2}(y_0)$ contains a periodic cube $Q_1(y_0)$. This implies that $\sup_{B_{\sqrt{n}/2}(y_0)} \hat{w}^\delta = \sup_{\mathbb{R}^n} \hat{w}^\delta$ and $\inf_{B_{\sqrt{n}/2}(y_0)} \hat{w}^\delta = \inf_{\mathbb{R}^n} \hat{w}^\delta = 0$. Now we apply the Harnack inequality over $B_{\sqrt{n}}(y_0)$ to (2.1.8) (see Theorem A.0.1 (c) with $f = \delta \min_{\mathbb{R}^n} w^\delta - a_{ij}M_{ij}$). Then

$$\sup_{B_{\sqrt{n}/2}(y_0)} \hat{w}^\delta \leq c_0 \left\| \lambda^{-1} (\delta \min_{\mathbb{R}^n} w^\delta - a_{ij}M_{ij}) \right\|_{L^\infty(B_{\sqrt{n}}(y_0))} \leq 2c_0 \lambda^{-1} n\sigma \|M\|;$$

here we utilized (2.1.4) and (2.1.5). Since the above bound is independent of $\delta \in (0, 1)$, and since y_0 is an arbitrary point, we have shown (2.1.6) with $C = 2c_0 \lambda^{-1} n\sigma$.

Define now

$$\tilde{w}^\delta(y) := w^\delta(y) - w^\delta(0) \quad (y \in \mathbb{R}^n).$$

By (2.1.6), $\|\tilde{w}^\delta\|_{L^\infty(\mathbb{R}^n)} \leq 4c_0 \lambda^{-1} n\sigma \|M\|$. Moreover, $\tilde{w}^\delta \in C^{2,\alpha}(\mathbb{R}^n)$ and satisfies

$$a_{ij}D_{y_i y_j} \tilde{w}^\delta + a_{ij}M_{ij} - \delta \tilde{w}^\delta = \delta w^\delta(0) \quad \text{in } \mathbb{R}^n.$$

As we did when proving (2.1.3), we apply an interior Schauder estimate (Proposition A.0.1 (b) with $\Omega = B_1(y_0)$ for $y_0 \in \mathbb{R}^n$) to the above equation, which yields that

$$\begin{aligned} \sup_{0 < \delta < 1} \|\tilde{w}^\delta\|_{C^{2,\alpha}(B_1(y_0))} &\leq c_1 \left(\|\tilde{w}^\delta\|_{L^\infty(B_1(y_0))} + \|a_{ij}M_{ij}\|_{C^\alpha(B_1(y_0))}^{(2)} + |\delta w^\delta(0)| \right) \\ &\leq \tilde{c}_1 c_0 n\sigma (\lambda^{-1} + 1) \|M\|, \end{aligned}$$

which verifies (2.1.7) with $C = \tilde{c}_1 c_0 n\sigma (\lambda^{-1} + 1)$. \square

Now we are ready to prove Lemma 2.1.2

Proof of Lemma 2.1.2. In view of (2.1.5), we can take a subsequence $\{\delta_k w^{\delta_k}(0)\}_{k=1}^\infty$ of $\{\delta w^\delta\}_{0 < \delta < 1}$ and a number $\gamma \in \mathbb{R}$ such that $\delta_k w^{\delta_k}(0) \rightarrow \gamma$ as $k \rightarrow \infty$. Then (2.1.6) implies that $\delta_k w^{\delta_k} \rightarrow \gamma$ uniformly in \mathbb{R}^n as $k \rightarrow \infty$.

On the other hand, (2.1.7) allows us to use the compact embedding result (see Theorem C.0.3), from which we deduce that there is $w \in C^{2,\alpha}(\mathbb{R}^n)$ and a further subsequence of $\{\delta_k\}_{k=1}^\infty$, which we denote again by $\{\delta_k\}_{k=1}^\infty$ for convenience, such that $\tilde{w}^{\delta_k} \rightarrow w$ with respect to $C^2(\mathbb{R}^n)$ -norm; i.e.,

$$(2.1.9) \quad \|\delta_k w^{\delta_k} - \gamma\|_{L^\infty(\mathbb{R}^n)} + \|\tilde{w}^{\delta_k} - w\|_{C^2(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

By the stability of viscosity solutions (c.f. Theorem B.0.2), w solves (2.1.1) in the viscosity sense. Then the $C^{2,\alpha}(\mathbb{R}^n)$ -regularity of w forces itself to be a classical solution.

To this end we prove that the constant γ is unique. Suppose to the contrary that there is another $\gamma' \in \mathbb{R}$ to which a subsequence of $\{\delta w^\delta\}_{0 < \delta < 1}$ converges uniformly in \mathbb{R}^n . Denote w' , which belongs to $C^{2,\alpha}(\mathbb{R}^n)$, by the corresponding limit of a (further) subsequence of $\{\tilde{w}^\delta\}_{0 < \delta < 1}$.

Assume without loss of generality that $\gamma < \gamma'$. As w and w' being bounded, we are able to add a constant t_0 to w in such a way that $w'(y_0) + t_0 < w(y_0)$ at a point $y_0 \in \mathbb{R}^n$. Take t_1 by the infimum value of t such that $w' + t \geq w$ in \mathbb{R}^n . Then $w' + t_1$ touches w by above at a point y_1 . Since w is a subsolution of (2.1.1),

$$\gamma \leq a_{ij}(y_1)D_{y_i y_j}(w' + t_1)(y_1) + a_{ij}(y_1)M_{ij} = a_{ij}(y_1)D_{y_i y_j}w'(y_1) + a_{ij}(y_1)M_{ij} = \gamma',$$

which is a contradiction. It shows that the constant γ must be unique.

We observe that the uniqueness of γ implies that the convergence (2.1.9) holds without extracting a subsequence. Assume that $\{\tilde{w}^{\delta_k}\}_{k=1}^\infty$ [resp. $\{\tilde{w}^{\delta'_k}\}_{k=1}^\infty$] converges to w [resp. w'] with respect to $\|\cdot\|_{C^2(\mathbb{R}^n)}$. By the uniqueness of γ , both w and w' satisfy (2.1.1). Thus, $w - w'$ solves $a_{ij}D_{y_i y_j}(w - w') = 0$ in \mathbb{R}^n . Since w and w' are bounded, it follows from Liouville's theorem (see Theorem Proposition A.0.1 (d)) that $w - w' \equiv t$ in \mathbb{R}^n for some constant t . However, the fact that $\tilde{w}^\delta(0) = 0$ for all $\delta \in (0, 1)$ implies $w(0) = w'(0) = 0$. Therefore, $w \equiv w'$ in \mathbb{R}^n .

The last assertion of Lemma 2.1.2 also follows from Liouville's theorem whose proof is identical to which is shown just above. \square

From now on we denote $w^\delta(\cdot; M)$ by the unique solution of (2.1.2) for a given $M \in \mathcal{S}^n$. Also

$$\tilde{w}^\delta(\cdot; M) = w^\delta(\cdot; M) - \min_{\mathbb{R}^n} w^\delta(\cdot; M), \quad \bar{w}^\delta(\cdot; M) = w^\delta(\cdot; M) - w^\delta(0; M).$$

In addition, let us write $w(\cdot; M)$ by the solution of (2.1.1) for a given $M \in \mathcal{S}^n$ which is normalized by 0; i.e., $w(0; M) = 0$.

By Lemma 2.1.2 we can understand γ as a functional $M \mapsto \gamma(M)$ on \mathcal{S}^n . The linear structure of the equation (2.1.1) allows us to obtain further information about the functional γ which is stated in the next lemma.

Lemma 2.1.5. *Let γ be the functional on \mathcal{S}^n obtained from Lemma 2.1.2. Then*

- (i) γ is linear on \mathcal{S}^n ; i.e., there is a constant matrix (\bar{a}_{ij}) such that $\gamma(M) = \bar{a}_{ij}M_{ij}$.
- (ii) The matrix (\bar{a}_{ij}) is symmetric and elliptic with the same ellipticity constants of (a_{ij}) ; i.e.,

$$\lambda|\xi|^2 \leq \bar{a}_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

Proof. Let $M, N \in \mathcal{S}^n$. Then the linear structure of (2.1.2) yields

$$\begin{aligned} & a_{ij}D_{y_i y_j}(w(\cdot; M) + c w(\cdot; N)) + a_{ij}(M_{ij} + c N_{ij}) - \delta(w(\cdot; M) + c w(\cdot; N)) \\ &= (a_{ij}D_{y_i y_j}w(\cdot; M) + a_{ij}M_{ij} - \delta w(\cdot; N)) + c(a_{ij}D_{y_i y_j}w(\cdot; N) + a_{ij}N_{ij} - \delta w(\cdot; N)) \\ &= 0 \quad \text{in } \mathbb{R}^n. \end{aligned}$$

By the uniqueness of the solution to (2.1.2), we conclude

$$w^\delta(\cdot; M + cN) = w^\delta(\cdot; M) + c w^\delta(\cdot; N).$$

The assertion (i) is hence accomplished by (2.1.9); by putting $\bar{a}_{ij} = \gamma(E^{ij})$, we obtain $\gamma(M) = \bar{a}_{ij}M_{ij}$. It also shows that $(\bar{a}_{ij}) \in \mathcal{S}^n$, which proves the first part of (ii).

We prove the second part of (ii). Choose any $\varepsilon > 0$ and assume for a contradiction that there exists $\xi \in \mathbb{R}^n$ for which $\bar{a}_{ij}\xi_i\xi_j < (\lambda - \varepsilon)|\xi|^2$. In view of (2.1.9), there corresponds $\delta \in (0, 1)$ for which

$$\left\| \delta w^\delta(\cdot; \xi \cdot \xi^t) - \bar{a}_{ij}\xi_i\xi_j \right\|_{L^\infty(\mathbb{R}^n)} < \varepsilon|\xi|^2.$$

For the moment we abbreviate $w^\delta(\cdot; \xi \cdot \xi^t)$ by w^δ . Then we have

$$a_{ij}D_{y_i y_j} w^\delta = \delta w^\delta - a_{ij}\xi_i\xi_j \leq \left\| \delta w^\delta - \bar{a}_{ij}\xi_i\xi_j \right\|_{L^\infty(\mathbb{R}^n)} + (\bar{a}_{ij}\xi_i\xi_j - \lambda|\xi|^2) < 0 \quad \text{in } \mathbb{R}^n,$$

which is contradictory to the fact that w^δ achieves a global minimum. Hence, we conclude that $\lambda \|N\| \leq \bar{a}_{ij}N_{ij}$ for any N with $N \geq 0$ which proves the first inequality in (ii).

A similar argument applies in proving the rightmost inequality in (ii). \square

Define the operator \bar{L} by

$$\bar{L}v = \bar{a}_{ij}D_{ij}v.$$

We call \bar{L} the *effective operator* in the sense of the following lemma. Recall from Lemma 2.1.1 the limit function u of a subsequence of $\{u^\varepsilon\}_{\varepsilon>0}$.

Lemma 2.1.6. *Suppose that (L_ε) satisfies the structure conditions (L1)-(L2) and let $\{u^\varepsilon\}_{\varepsilon>0} \subset C^{m+2,\alpha}(\Omega)$ be the family of solutions to (L_ε) . Then there exists a unique function u , which has a modulus of continuity on $\bar{\Omega}$, such that $u^\varepsilon \rightarrow u$ uniformly in $\bar{\Omega}$ as $\varepsilon \rightarrow 0$. Moreover, $u \in C^{m+2,\alpha}(\bar{\Omega})$ and it solves*

$$(\bar{L}) \quad \begin{cases} \bar{a}_{ij}D_{ij}u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

Proof. We already proved part of the first assertion in Lemma 2.1.1. Since $u^\varepsilon \rightarrow u$ uniformly in $\bar{\Omega}$ up to a subsequence and $u^\varepsilon = g$ on $\partial\Omega$ for all $\varepsilon > 0$, we have $u = g$ on $\partial\Omega$.

We claim that u is a viscosity solution to (\bar{L}) . Suppose that this claim is true. Since (\bar{a}_{ij}) is a constant positive definite matrix, by a change of coordinate (\bar{L}) transforms into a Laplace equation. This implies that $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Then we apply Theorem Proposition A.0.1 (e) with $k = m$ and obtain $u \in C^{m+2,\alpha}(\bar{\Omega})$.

On the other hand, the maximum principle implies that (\bar{L}) has at most one solution. Therefore, the convergence of $u^\varepsilon \rightarrow u$ is valid without extracting a subsequence.

Thus, we are only left with proving the above claim. Let P be a paraboloid which touches u by above at x_0 in a neighborhood. By replacing P by $P + \eta|x - x_0|^2$ ($\eta > 0$) we may assume that P touches u strictly by above. Assume to the contrary that

$$\bar{a}_{ij}D_{ij}P - f(x_0) < 0.$$

By the continuity of f , we can choose $r > 0$ in such a way that $B_r(x_0) \subset \Omega$ and

$$\bar{a}_{ij}D_{ij}P - f(x) < 0, \quad \forall x \in B_r(x_0).$$

Define

$$P^\varepsilon(x) = P(x) + \varepsilon^2 w\left(\frac{x}{\varepsilon}; D^2P\right).$$

Note that $P^\varepsilon \in C^{2,\alpha}(\overline{\Omega})$. In view of (2.1.1) we obtain

$$a_{ij} \left(\frac{x}{\varepsilon} \right) D_{ij} P^\varepsilon(x) - f(x) = \bar{a}_{ij} D_{ij} P - f(x) < 0$$

in $B_r(x_0)$. Hence, P^ε is a supersolution of (L_ε) so that the strong maximum principle implies

$$(u^\varepsilon - P^\varepsilon)(x_0) < \max_{\partial B_r(x_0)} (u^\varepsilon - P^\varepsilon),$$

Letting $\varepsilon \rightarrow 0$ then gives

$$0 = (u - P)(x_0) \leq \max_{\partial B_r(x_0)} (u - P),$$

which violates the assumption that P touches u strictly by above at x_0 . Therefore, $\bar{a}_{ij} D_{ij} P - f(x) \geq 0$ for any $x \in \Omega$. It shows that u is a viscosity subsolution of (\bar{L}) .

In a similar manner, we are able to prove that u is a viscosity supersolution of (\bar{L}) . This completes the proof. \square

2.2. Interior and boundary layer correctors. In this subsection, we seek the interior and boundary layer correctors. We make a remark from the previous section before we begin. Recall from the linear algebra, $\{E^{ij} | i, j = 1, \dots, n\}$ is the standard basis of \mathcal{S}^n . Any matrix $M \in \mathcal{S}^n$ can be written as $M = M_{ij} E^{ij}$ where $M = (M_{ij})$. Set $M = E^{kl}$ in Lemma 2.1.2 for $k, l \in \{1, \dots, n\}$ and write $\chi^{kl} = w(\cdot; E^{kl}) \in C^{2,\alpha}(\mathbb{R}^n)$. Notice that $\chi^{kl}(0) = 0$. In view of (2.1.1) and Lemma 2.1.5 (i), χ^{kl} solves

$$(2.2.1) \quad a_{ij} D_{ij} \chi^{kl} + a_{kl} = \bar{a}_{kl}.$$

Multiplying (2.2.1) with M_{kl} and summing over the indices $k, l = 1, \dots, n$, we see that $\chi^{kl} M_{kl}$ solves (2.1.1) with $M = (M_{kl})$. Define

$$w_2(y, x) = \chi^{kl}(y) D_{x_k x_l} u(x) + \psi_2(x) \quad (y \in \mathbb{R}^n, x \in \overline{\Omega}),$$

where u is given by Lemma 2.1.6 and ψ_2 is chosen arbitrarily from $C^{m,\alpha}(\overline{\Omega})$ for the moment. By Lemma 2.1.6, $w(\cdot, x) \in C^{2,\alpha}(\mathbb{R}^n)$ for each $x \in \overline{\Omega}$ while $w(y, \cdot) \in C^{m,\alpha}(\overline{\Omega})$ for each $y \in \mathbb{R}^n$. Moreover, $w_2(\cdot, x)$ solves

$$a_{ij} D_{y_i y_j} w_2(\cdot, x) + a_{ij} D_{x_i x_j} u(x) = 0 \quad \text{in } \mathbb{R}^n$$

for each $x \in \overline{\Omega}$. We call w_2 the second order (interior) corrector of (L_ε) .

Interior correctors of higher orders are discovered in the similar direction.

Lemma 2.2.1. *There are a family $\{\bar{a}_{i_1 \dots i_k} | 1 \leq i_1, \dots, i_k \leq n, k \geq 2\}$ of constants and a family $\{\chi^{i_1 \dots i_k} | 1 \leq i_1, \dots, i_k \leq n, k \geq 2\}$ of 1-periodic functions in $C^{2,\alpha}(\mathbb{R}^n)$ which satisfy the following recursive equations*

$$(2.2.2) \quad a_{ij} D_{ij} \chi^{i_1 \dots i_k} + 2a_{i_k j} D_{y_j} \chi^{i_1 \dots i_{k-1}} + a_{i_{k-1} i_k} \chi^{i_1 \dots i_{k-2}} = \bar{a}_{i_1 \dots i_k} \quad \text{in } \mathbb{R}^n$$

for each $1 \leq i_1, \dots, i_k \leq n$. Here we understand $\chi \equiv 1$ and $\chi^i \equiv 0$ for each $i = 1, \dots, n$. Furthermore, for each $k \geq 2$, $\chi^{i_1 \dots i_k}(0) = 0$ and

$$(2.2.3) \quad |\bar{a}_{i_1 \dots i_k}| + \|\chi^{i_1 \dots i_k}\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_k, \quad \forall 1 \leq i_1, \dots, i_k \leq n.$$

Proof. We already know $\{\bar{a}_{ij}\}_{i,j=1,\dots,n}$ and $\{\chi^{ij}\}_{i,j=1,\dots,n}$ from the comment above this lemma; one may notice that (2.2.2) is exactly the same with (2.2.1) if $k = 2$. Also note from (2.1.3) and (2.1.7) that

$$|\bar{a}_{ij}| + \|\chi^{ij}\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C =: C_2,$$

where K_1 and K_2 are chosen as in Lemma 2.1.3 and 2.1.4 respectively.

The construction of the families $\{\bar{a}_{i_1 \dots i_k}\}$ and $\{\chi^{i_1 \dots i_k}\}$ (for $k \geq 3$) can be done by an induction argument. The main part of this proof follows the same line of the proof of Lemma 2.1.2; for the sake of completeness, let us present the proof.

Let $k \geq 3$ and assume that we have already found the families $\{\bar{a}_{i_1 \dots i_l}\}$ and $\{\chi^{i_1 \dots i_l}\}$ for $l < k$. Fix the indices $1 \leq i_1, \dots, i_k \leq n$ and consider the approximated problem

$$(2.2.4) \quad a_{ij} D_{y_i y_j} v^\delta + 2a_{ikj} D_{y_j} \chi^{i_1 \dots i_{k-1}} + a_{i_{k-1} i_k} \chi^{i_1 \dots i_{k-2}} - \delta v^\delta = 0 \quad \text{in } \mathbb{R}^n.$$

Note that this equation belongs to the same class of (2.1.2). Moreover,

$$\left\| 2a_{ikj} D_{y_j} \chi^{i_1 \dots i_{k-1}} + a_{i_{k-1} i_k} \chi^{i_1 \dots i_{k-2}} \right\|_{C^\alpha(\mathbb{R}^n)} \leq (2C_{k-1} + C_{k-2})n\sigma.$$

Hence, Lemma 2.1.3 holds. To be specific, there exists a unique bounded 1-periodic solution v^δ to (2.2.4) for each $\delta \in (0, 1)$; in addition, $v^\delta \in C^{2,\alpha}(\mathbb{R}^n)$ satisfying

$$(2.2.5) \quad \sup_{0 < \delta < 1} \left\| \delta v^\delta \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq c_0(2C_{k-1} + C_{k-2}).$$

Now we can apply Lemma 2.1.4, which yields

$$(2.2.6) \quad \sup_{0 < \delta < 1} \left\| \tilde{v}^\delta \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq c_1(2C_{k-1} + C_{k-2}).$$

To this end, we are ready to assert as in the proof of Lemma 2.1.2. As a result, we obtain a unique number $\bar{a}_{i_1 \dots i_k}$ and a 1-periodic function $\chi^{i_1 \dots i_k} \in C^{2,\alpha}(\mathbb{R}^n)$ such that the pair $(\bar{a}_{i_1 \dots i_k}, \chi^{i_1 \dots i_k})$ solves (2.2.2) and

$$(2.2.7) \quad \left\| \delta v^\delta - \bar{a}_{i_1 \dots i_k} \right\|_{L^\infty(\mathbb{R}^n)} + \left\| \tilde{v}^\delta - \chi^{i_1 \dots i_k} \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0.$$

Note that $\chi^{i_1 \dots i_k}(0) = 0$. In addition, we get (2.2.3) with $C_k = c_1(2C_{k-1} + C_{k-2})$, in view of (2.2.5) and (2.2.6). \square

Now let $m \geq 3$. By Lemma 2.1.6 we have $u \in C^{m+2,\alpha}(\bar{\Omega})$. For $1 \leq k \leq m-2$, define $\psi_k \in C^{m-k+2,\alpha}(\bar{\Omega})$ recursively by the unique solution of

$$(2.2.8) \quad \begin{cases} \bar{a}_{ij} D_{x_i x_j} \psi_k = - \sum_{l=3}^{k+2} \bar{a}_{i_1 \dots i_l} D_{x_{i_1} \dots x_{i_l}} \psi_{k-l+2} & \text{in } \Omega, \\ \psi_k = 0 & \text{on } \partial\Omega, \end{cases}$$

where we understand $\psi_0 = u$. This can be done by an induction argument. Fix k and suppose that $\psi_l \in C^{m-l+2,\alpha}(\bar{\Omega})$ for all $0 \leq l < k$. Then the right hand side of (2.2.8) belongs to $C^{m-k,\alpha}(\bar{\Omega})$. Now the existence and regularity theories (Proposition A.0.1 (e) and (f)) ensure that the boundary value problem (2.2.8) attains a unique solution $\psi_k \in C^{m-k+2,\alpha}(\bar{\Omega})$. This induction holds because the induction hypothesis is met for $k = 1$.

Furthermore, we have the following.

Lemma 2.2.2. *Let $m \geq 3$ and set ψ_k as above for $1 \leq k \leq m-2$. Then*

$$(2.2.9) \quad \left\| \psi_k \right\|_{C^{m-k+2,\alpha}(\bar{\Omega})} \leq \tilde{C}_{k,m,\Omega} \left(\left\| f \right\|_{C^{m,\alpha}(\bar{\Omega})} + \left\| g \right\|_{C^{m+2,\alpha}(\bar{\Omega})} \right),$$

for each $k = 0, 1, \dots, m-2$, where we understand $\psi_0 = u$.

Proof. Since $u \in C^{m+2,\alpha}(\bar{\Omega})$ solves (\bar{L}) and since $f \in C^{m,\alpha}(\bar{\Omega})$, $g \in C^{m+2,\alpha}(\bar{\Omega})$ and $\partial\Omega \in C^{m+2,\alpha}$, Theorem Proposition A.0.1 (e) applies so that

$$\left\| u \right\|_{C^{m+2,\alpha}(\bar{\Omega})} \leq C_{m,\Omega} \left(\left\| u \right\|_{L^\infty(\Omega)} + \left\| f \right\|_{C^{m,\alpha}(\bar{\Omega})} + \left\| g \right\|_{C^{m+2,\alpha}(\bar{\Omega})} \right).$$

Now the a priori estimate to (\bar{L}) (Proposition A.0.1 (a)) yields

$$\|u\|_{C^{m+2,\alpha}(\bar{\Omega})} \leq C'_{m,\Omega} (\|f\|_{C^{m,\alpha}(\bar{\Omega})} + \|g\|_{C^{m+2,\alpha}(\bar{\Omega})}).$$

The proof is finished by adopting an induction argument with

$$\tilde{C}_{k,m,\Omega} = C'_{m-k+2,\Omega} \left(\sum_{l=3}^{k+2} C_l \tilde{C}_{k-l+2,m,\Omega} \right).$$

□

Set for each $1 \leq k \leq m$

$$(2.2.10) \quad w_k(y, x) = \sum_{l=1}^k \chi^{i_1 \dots i_l}(y) D_{x_{i_1} \dots x_{i_l}} \psi_{k-l}(x) + \psi_k(x) \quad (y \in \mathbb{R}^n, x \in \bar{\Omega}),$$

where $\psi_{m-1} \in C^{3,\alpha}(\bar{\Omega})$ and $\psi_m \in C^{2,\alpha}(\bar{\Omega})$ are arbitrary functions which satisfy (2.2.9) respectively when $k = m-1$ and m . Recall that we have set $\chi^i \equiv 0$ for all $i = 1, \dots, n$, which implies that $w_1(y, x) = \psi_1(x)$; that is, w_1 is independent of the y -variable. Additionally, $w_k(\cdot, x) \in C^{2,\alpha}(\mathbb{R}^n)$ for each $x \in \bar{\Omega}$ and $w_k(y, \cdot) \in C^{m-k+2,\alpha}(\bar{\Omega})$ for each $y \in \mathbb{R}^n$. By virtue of (2.2.3) and (2.2.9), we observe that

$$\sup_{x \in \bar{\Omega}} \|w_k(\cdot, x)\|_{C^{2,\alpha}(\mathbb{R}^n)} + \sup_{y \in \mathbb{R}^n} \|w_k(y, \cdot)\|_{C^{m-k+2,\alpha}(\bar{\Omega})} \leq \tilde{C}_{k,m,\Omega} (\|f\|_{C^{m,\alpha}(\bar{\Omega})} + \|g\|_{C^{m+2,\alpha}(\bar{\Omega})}),$$

where $\tilde{C}_{k,m,\Omega} = \sum_{l=1}^k n^l C_l \tilde{C}_{k-l,m,\Omega} + \tilde{C}_{k,m,\Omega}$ for each $k = 1, \dots, m$.

Lemma 2.2.3. *Let $m \geq 3$ be an integer and w_k be given by (2.2.10) for each $k = 1, \dots, m$. Then for $3 \leq k \leq m$, w_k solves recursively*

$$(2.2.11) \quad a_{ij} D_{y_i y_j} w_k + 2a_{ij} D_{x_i y_j} w_{k-1} + a_{ij} D_{x_i x_j} w_{k-2} = 0 \quad \text{in } \mathbb{R}^n \times \bar{\Omega}.$$

Proof. In view of (2.2.2) and (2.2.8) we observe that

$$\begin{aligned} & a_{ij} D_{y_i y_j} w_k + 2a_{ij} D_{x_i y_j} w_{k-1} + a_{ij} D_{x_i x_j} w_{k-2} \\ &= \sum_{l=3}^k \left(a_{ij} D_{y_i y_j} \chi^{i_1 \dots i_l} + 2a_{ij} D_{y_j} \chi^{i_1 \dots i_{l-1}} + a_{i_{l-1} i_l} \chi^{i_1 \dots i_{l-2}} \right) D_{x_{i_1} \dots x_{i_l}} \psi_{k-l} + a_{ij} D_{x_i x_j} \psi_{k-2} \\ &= \sum_{l=3}^k \bar{a}_{i_1 \dots i_l} D_{x_{i_1} \dots x_{i_l}} \psi_{k-l} + a_{ij} D_{x_i x_j} \psi_{k-2} = 0 \end{aligned}$$

in $\mathbb{R}^n \times \bar{\Omega}$, for each $k = 3, \dots, m$. □

Define now the k -th order interior corrector w_k^ε of (L_ε) for each $1 \leq k \leq m$ and $\varepsilon > 0$ by

$$w_k^\varepsilon(x) = w_k\left(\frac{x}{\varepsilon}, x\right) \quad (x \in \bar{\Omega}),$$

and accordingly, the k -th order boundary layer corrector z_k^ε of (L_ε) by the solution of

$$(2.2.12) \quad \begin{cases} a_{ij} \left(\frac{x}{\varepsilon}\right) D_{ij} z_k^\varepsilon = 0 & \text{in } \Omega, \\ z_k^\varepsilon = -w_k^\varepsilon & \text{on } \partial\Omega. \end{cases}$$

To specify the definitions of w_k^ε and z_k^ε , we fix $\varepsilon > 0$ and $k \in \{1, \dots, m\}$. From the comment above Lemma 2.2.3 we know that $w_k^\varepsilon \in C^{2,\alpha}(\bar{\Omega})$. Moreover, the coefficient

$a_{ij}(\cdot/\varepsilon)$ lies in $C^\alpha(\overline{\Omega})$ and uniformly elliptic over $\overline{\Omega}$ (with the constants λ and Λ). Hence, the existence and regularity theories (Proposition A.0.1 (e) and (f)) ascertain that there is a unique $C^{2,\alpha}(\overline{\Omega})$ solution, which we denote by z_k^ε , for (2.2.12).

Now let us investigate the uniform estimate of z_k^ε with respect to $\varepsilon > 0$. The $C^{2,\alpha}$ norm of w_k^ε is no longer uniformly bounded on $\overline{\Omega}$; nevertheless its L^∞ -norm is bounded uniformly with respect to ε :

$$(2.2.13) \quad \sup_{\varepsilon > 0} \|w_k^\varepsilon\|_{L^\infty(\overline{\Omega})} \leq \sum_{l=1}^k \sum_{1 \leq i_1, \dots, i_l \leq n} \|\chi^{i_1 \dots i_l}\|_{L^\infty(\Omega)} \|\psi_{k-l}\|_{C^l(\overline{\Omega})} + \|\psi_k\|_{L^\infty(\Omega)} \\ \leq \bar{C}_{k,m,\Omega} (\|f\|_{C^{m,\alpha}(\overline{\Omega})} + \|g\|_{C^{m+2,\alpha}(\overline{\Omega})}).$$

As we pointed out in the proof of Lemma 2.1.1, the ellipticity constants of $(a_{ij}(\cdot/\varepsilon))$ are fixed by λ and Λ with respect to the change of $\varepsilon > 0$. Hence, there is an a priori estimate (Proposition A.0.1 (a)) such that

$$(2.2.14) \quad \sup_{\varepsilon > 0} \|z_k^\varepsilon\|_{L^\infty(\Omega)} \leq c_0 \|w_k^\varepsilon\|_{L^\infty(\Omega)} \leq c_0 \bar{C}_{k,m,\Omega} (\|f\|_{C^{m,\alpha}(\overline{\Omega})} + \|g\|_{C^{m+2,\alpha}(\overline{\Omega})}).$$

Note that for any $\varepsilon > 0$, $z_1^\varepsilon \equiv 0$ on $\overline{\Omega}$, since $w_1^\varepsilon \equiv \psi_1$ on $\overline{\Omega}$ where ψ_1 vanishes on $\partial\Omega$.

2.3. Proof of Main Theorem I. We are now in position to prove Main Theorem I.

Proof of Theorem 1.1.1. Fix $\varepsilon > 0$. Let w_k^ε and z_k^ε be defined as in the previous section for each $k = 1, \dots, m$. Define

$$\eta_m^\varepsilon = u + \varepsilon w_1^\varepsilon + \varepsilon^2 w_2^\varepsilon + \dots + \varepsilon^m w_m^\varepsilon, \quad \theta_m^\varepsilon = \varepsilon z_1^\varepsilon + \varepsilon^2 z_2^\varepsilon + \dots + \varepsilon^m z_m^\varepsilon$$

on $\overline{\Omega}$. Then both η_m^ε and θ_m^ε belong to $C^{2,\alpha}(\overline{\Omega})$. By means of (2.1.1), (2.2.11) and (2.2.12) as well as by noting that

$$D^2 w_k^\varepsilon = (\varepsilon^{-2} D_y^2 + \varepsilon^{-1} (D_x D_y + D_y D_x) + D_x^2) w_k \left(\frac{\cdot}{\varepsilon}, \cdot \right),$$

we obtain

$$\begin{aligned} a_{ij} \left(\frac{x}{\varepsilon} \right) D_{ij} (\eta_m^\varepsilon + \theta_m^\varepsilon) &= a_{ij} \left(\frac{x}{\varepsilon} \right) D_{ij} \eta_m^\varepsilon \\ &= a_{ij} \left(\frac{x}{\varepsilon} \right) D_{y_i y_j} w_2 \left(\frac{x}{\varepsilon}, x \right) + a_{ij} \left(\frac{x}{\varepsilon} \right) D_{x_i x_j} u(x) \\ &\quad + \sum_{k=3}^m \varepsilon^{k-2} \left[a_{ij} \left(\frac{x}{\varepsilon} \right) D_{y_i y_j} w_k \left(\frac{x}{\varepsilon}, x \right) + 2a_{ij} \left(\frac{x}{\varepsilon} \right) D_{x_i y_j} w_{k-1} \left(\frac{x}{\varepsilon}, x \right) + a_{ij} \left(\frac{x}{\varepsilon} \right) D_{x_i x_j} w_{k-2} \left(\frac{x}{\varepsilon}, x \right) \right] \\ &\quad + \varepsilon^{m-1} \left[a_{ij} \left(\frac{x}{\varepsilon} \right) D_{x_i x_j} w_{m-1} \left(\frac{x}{\varepsilon}, x \right) + 2a_{ij} \left(\frac{x}{\varepsilon} \right) D_{x_i y_j} w_m \left(\frac{x}{\varepsilon}, x \right) + \varepsilon a_{ij} \left(\frac{x}{\varepsilon} \right) D_{x_i x_j} w_m \left(\frac{x}{\varepsilon}, x \right) \right] \\ &= f + \varepsilon^{m-1} \left[a_{ij} \left(\frac{x}{\varepsilon} \right) D_{x_i x_j} w_{m-1} \left(\frac{x}{\varepsilon}, x \right) + 2a_{ij} \left(\frac{x}{\varepsilon} \right) D_{x_i y_j} w_m \left(\frac{x}{\varepsilon}, x \right) + \varepsilon a_{ij} \left(\frac{x}{\varepsilon} \right) D_{x_i x_j} w_m \left(\frac{x}{\varepsilon}, x \right) \right] \\ &= f + \varepsilon^{m-1} \varphi_m^\varepsilon \end{aligned}$$

in Ω , where

$$\begin{aligned} \varphi_m^\varepsilon(x) &= \sum_{l=2}^{m-1} \left[2a_{i_l j} \left(\frac{x}{\varepsilon} \right) D_{y_j} \chi^{i_1 \dots i_{l-1}} \left(\frac{x}{\varepsilon} \right) + a_{i_{l-1} i_l} \left(\frac{x}{\varepsilon} \right) \chi^{i_1 \dots i_{l-2}} \left(\frac{x}{\varepsilon} \right) \right] D_{x_{i_1} \dots x_{i_l}} \psi_{m-l-1}(x) \\ &\quad + \varepsilon \sum_{l=2}^m a_{i_{l-1} i_l} \left(\frac{x}{\varepsilon} \right) \chi^{i_1 \dots i_{l-2}} \left(\frac{x}{\varepsilon} \right) D_{x_{i_1} \dots x_{i_l}} \psi_{m-l}(x) \quad (x \in \Omega). \end{aligned}$$

Now we set $\varepsilon \in (0, 1)$. According to (2.2.3) and (2.2.9), we have

$$\|\varphi_m^\varepsilon\|_{L^\infty(\Omega)} \leq L_{m,\Omega} \left(\|f\|_{C^{m,\alpha}(\overline{\Omega})} + \|g\|_{C^{m+2,\alpha}(\overline{\Omega})} \right)$$

where

$$L_{m,\Omega} = \sigma \left[\sum_{l=3}^{m-1} n^{l-1} \left\{ 2(C_{l-1} + C_{l-2})\tilde{C}_{m-l-1,\Omega} + C_{l-2}\tilde{C}_{m-l,\Omega} \right\} + n^{m-1}C_{m-2}\tilde{C}_{0,\Omega} \right].$$

Here C_k and \tilde{C}_k are the constants chosen as in (2.2.3) and (2.2.9).

On the other hand, we have

$$\eta_m^\varepsilon + \theta_m^\varepsilon = g + \sum_{k=1}^m \varepsilon^k (w_k^\varepsilon + z_k^\varepsilon) = g \quad \text{on } \partial\Omega.$$

Thus,

$$\begin{cases} a_{ij}\left(\frac{x}{\varepsilon}\right) D_{ij}(u^\varepsilon - \eta_m^\varepsilon - \theta_m^\varepsilon) = -\varepsilon^{m-1}\varphi_m^\varepsilon & \text{in } \Omega, \\ u^\varepsilon - \eta_m^\varepsilon - \theta_m^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

As $u^\varepsilon - \eta_m^\varepsilon - \theta_m^\varepsilon \in C^{2,\alpha}(\overline{\Omega})$, it follows from an a priori estimate (Proposition A.0.1 (a)) that

$$\|u^\varepsilon - \eta_m^\varepsilon - \theta_m^\varepsilon\|_{L^\infty(\Omega)} \leq c_0 \varepsilon^{m-1} \|\varphi_m^\varepsilon\|_{L^\infty(\Omega)} \leq c_0 L_m \left(\|f\|_{C^{m,\alpha}(\overline{\Omega})} + \|g\|_{C^{m+2,\alpha}(\overline{\Omega})} \right).$$

□

3. FULLY NONLINEAR EQUATIONS IN NON-DIVERGENCE FORM

3.1. Basic homogenization scheme. This subsection is devoted to the homogenization process of (F_ε) to (\overline{F}) . It generalizes the homogenization result of linear equations (see Section 2.1). We highlight the fact that all the arguments carried out in this subsection are valid only with the assumption when F is locally Lipschitz continuous in $\mathcal{S}^n \times \overline{\Omega} \times \mathbb{R}^n$, i.e., when the structure conditions of F holds only for $m = 0$. We also note that one may find a general argument in [E2] for some lemmas. However, we present all the proofs which are adequate for our situation.

Lemma 3.1.1. *Assume for each $\varepsilon > 0$ that $u^\varepsilon \in C(\overline{\Omega})$ is a viscosity solution of (F_ε) . Then there is a function $u \in C(\overline{\Omega})$ and a subsequence $\{u^{\varepsilon_k}\}_{k=1}^\infty$ of $\{u^\varepsilon\}_{\varepsilon>0}$ such that $u^{\varepsilon_k} \rightarrow u$ uniformly in $\overline{\Omega}$ as $k \rightarrow \infty$.*

Proof. The proof is identical to that of Lemma 3.1.1. One may notice that the proof of Lemma 3.1.1 does not involve the linear structure of (L_ε) . □

As we did in Section 2.1, we will ascertain the effective equation which u solves in the viscosity sense at the end of this section.

Before we start, we point out that the argument throughout this subsection is valid by only assuming that $F \in C^{0,1}(\overline{B}_L \times \overline{\Omega} \times \mathbb{R}^n)$ for each $L > 0$ ((F3) with $m = 0$).

Lemma 3.1.2. *To each $(M, x) \in \mathcal{S}^n \times \overline{\Omega}$ there corresponds a unique number γ for which the following equation*

$$(3.1.1) \quad F(D_y^2 w + M, x, y) = \gamma \quad \text{in } \mathbb{R}^n$$

attains a 1-periodic solution $w \in C^{2,\alpha}(\mathbb{R}^n)$. Moreover, w is unique up to an additive constant. Moreover, if the solution w satisfies $w(0) = 0$, then

$$(3.1.2) \quad \|w\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{\|M\|}.$$

As we did in the linear case, we start with an approximating problem.

Lemma 3.1.3. *Let $(M, x) \in \mathcal{S}^n \times \overline{\Omega}$ and $\delta \in (0, 1)$. Then there is a unique bounded 1-periodic function $w^\delta \in C^{2,\alpha}(\mathbb{R}^n)$ which solves*

$$(3.1.3) \quad F(D_y^2 w^\delta + M, x, y) - \delta w^\delta = 0 \quad \text{in } \mathbb{R}^n,$$

with the uniform estimate

$$(3.1.4) \quad \sup_{0 < \delta < 1} \|\delta w^\delta\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{\|M\|}.$$

Proof. Fix $(M, x) \in \mathcal{S}^n \times \overline{\Omega}$. Using the structure conditions (F1)-(F4), one can easily verify that $G(N, r, y) = F(M + N, x, y) - \delta r$ has a comparison principle (see Theorem B.0.1 (a)).

We see that $w_-^\delta = -\delta^{-1}\sigma(1 + \|M\|)$ and similarly $w_+^\delta = \delta^{-1}\sigma(1 + \|M\|)$ are respectively a sub- and a supersolution of (3.1.3) for each δ . Thus, by Perron's method (Theorem B.0.1 (b) with $G(N, r, y) = F(M + N, x, y) - \delta r$, $u_- = w_-^\delta$ and $u_+ = w_+^\delta$), there is a unique bounded 1-periodic viscosity solution w^δ to (3.1.3) such that $w_-^\delta \leq w^\delta \leq w_+^\delta$ in \mathbb{R}^n . It implies

$$(3.1.5) \quad \sup_{0 < \delta < 1} \|\delta w^\delta\|_{L^\infty(\mathbb{R}^n)} \leq \sigma(1 + \|M\|).$$

Now we boost the regularity of w^δ to $C^{2,\alpha}(\mathbb{R}^n)$. Note that as long as we have the existence of w^δ , we may treat w^δ as the solution of $F(D^2 v + M, x, y) = f(y)$ where $f = \delta w^\delta$. Then the idea is to prove a uniform-in- δ Hölder regularity of δw^δ and Hölder regularity of F in the space variable, so that we may apply the interior $C^{2,\alpha}$ regularity theory (i.e., Theorem B.0.2 (e)).

Let $(M, x) \in \mathcal{S}^n \times \overline{\Omega}$ and define $G(N, y) = F(M + N, x, y)$ on $\mathcal{S}^n \times \mathbb{R}^n$. Let us also fix $y_0 \in \mathbb{R}^n$. According to (F4), G is concave in N . To measure the oscillation of G around y_0 , we consider

$$\tilde{\beta}_G(y, y_0) := \sup_{N \in \mathcal{S}^n} \frac{|G(N, y) - G(N, y_0)|}{1 + \|N\|} = \sup_{N \in \mathcal{S}^n} \frac{|F(M + N, x, y) - F(M + N, x, y_0)|}{1 + \|N\|}.$$

Then from (F3) we get that for any $y \in \mathbb{R}^n$

$$(3.1.6) \quad \tilde{\beta}_G(y, y_0) \leq \sigma(1 + \|M\|)|y - y_0|.$$

First we observe that $\delta w^\delta \in C^{\tilde{\alpha}}(\mathbb{R}^n)$ where $0 < \tilde{\alpha} < 1$ depends only on n, λ and Λ whose $C^{\tilde{\alpha}}$ norm is uniformly bounded for δ . Since w^δ is a solution to $G(D^2 v, y) = f(y)$ in \mathbb{R}^n , we have $w^\delta \in S(\lambda/n, \Lambda, \delta w^\delta - F(M, x, \cdot))$ in \mathbb{R}^n . As we restrict ourselves to the cube Q_2 , we obtain from Theorem B.0.2 (b) that $w^\delta \in C^{\tilde{\alpha}}(\overline{Q_1})$ and

$$\|w^\delta\|_{C^{\tilde{\alpha}}(\overline{Q_1})} \leq c_0 \left(\|w^\delta\|_{L^\infty(Q_2)} + \|\delta w^\delta - F(M, x, \cdot)\|_{L^\infty(Q_2)} \right) \leq c_0(\delta^{-1} + 2)\sigma(1 + \|M\|).$$

Since Q_1 is a periodic cube of w^δ , we obtain a uniform Hölder estimate on δw^δ over \mathbb{R}^n , namely,

$$(3.1.7) \quad \sup_{0 < \delta < 1} \|\delta w^\delta\|_{C^{\tilde{\alpha}}(\mathbb{R}^n)} \leq 3c_0\sigma(1 + \|M\|).$$

Now Theorem B.0.2 (e) applies to w^δ so that we get a constant $C_{\|M\|} > 1$ for which $w^\delta \in C^{2,\alpha}(\bar{B}_{C_{\|M\|}^{-1}\sqrt{n}}(y_0))$ and

$$\|w^\delta\|_{C^{2,\alpha}(\bar{B}_{C_{\|M\|}^{-1}\sqrt{n}}(y_0))}^* \leq C_{\|M\|} \left\{ \|w^\delta\|_{L^\infty(B_{\sqrt{n}}(y_0))} + 1 \right\} \leq \tilde{C}_{\|M\|} \delta^{-1},$$

where $\|\cdot\|_{C^{2,\alpha}(E)}^*$ is the adimensional $C^{2,\alpha}$ norm on E . Since $y_0 \in \mathbb{R}^n$ was an arbitrary point and $B_{\sqrt{n}}(y_0)$ contains a periodic cube of w^δ , we end up with (3.1.4). \square

Our next step is to find a uniform bound of the oscillation of w^δ for $\delta \in (0, 1)$.

Lemma 3.1.4. *Let $M \in \mathcal{S}^n$, $x \in \bar{\Omega}$ and w^δ be the unique solution to (3.1.3). Then*

$$(3.1.8) \quad \sup_{0 < \delta < 1} \operatorname{osc}_{\mathbb{R}^n} w^\delta \leq C(1 + \|M\|).$$

Moreover, there holds

$$(3.1.9) \quad \sup_{0 < \delta < 1} \|\tilde{w}^\delta\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{\|M\|},$$

where $\tilde{w}^\delta := w^\delta - w^\delta(0)$ in \mathbb{R}^n .

Proof. The proof follows the line of the proof of Lemma 2.1.4 instead of using Theorem B.0.2 (a) rather than Proposition A.0.1 (c). \square

It is noteworthy to observe that the derivatives of w^δ are bounded independent of $\delta \in (0, 1)$. To be specific, since $Dw^\delta = D\tilde{w}^\delta$ and $D^2w^\delta = D^2\tilde{w}^\delta$, we obtain from (3.1.9) that

$$(3.1.10) \quad \sup_{0 < \delta < 1} \left(\|Dw^\delta\|_{L^\infty(\mathbb{R}^n)} + \|D^2w^\delta\|_{L^\infty(\mathbb{R}^n)} + [D^2w^\delta]_{C^\alpha(\mathbb{R}^n)} \right) \leq C_{\|M\|}.$$

We are now in position to prove Lemma 3.1.2.

Proof of Lemma 3.1.2. One may notice that the proof of Lemma 2.1.2 has nothing to do with the linear structure of (2.1.1). Indeed, (3.1.5) and (3.1.9) respectively correspond to (2.1.5) and (2.1.7). Hence, Theorem C.0.3 allow us to extract a subsequence $\{\delta_k w^{\delta_k}, \tilde{w}^{\delta_k}\}_{k=1}^\infty$ from $\{\delta w^\delta, \tilde{w}^\delta\}_{0 < \delta < 1}$ such that

$$(3.1.11) \quad \|\delta_k w^{\delta_k} - \gamma\|_{L^\infty(\mathbb{R}^n)} + \|\tilde{w}^{\delta_k} - w\|_{C^2(\mathbb{R}^n)} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty,$$

for some $\gamma \in \mathbb{R}$ and $w \in C^{2,\alpha}(\mathbb{R}^n)$. In addition, we have that

$$|\gamma| = \lim_{k \rightarrow \infty} \|\delta_k w^{\delta_k}\|_{L^\infty(\mathbb{R}^n)} \leq \sigma(1 + \|M\|) \quad \text{and} \quad \|w\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{\|M\|}.$$

The rest of the proof is exactly the same with that of Lemma 2.1.2, except for using Corollary Theorem B.0.2 (c) as the Liouville theorem instead of Corollary Proposition A.0.1 (d). It can be done in the following way. Suppose that w and w' are two bounded 1-periodic $C^{2,\alpha}(\mathbb{R}^n)$ -solutions of (3.1.1) with the same constant γ . Then $w - w' \in S(\lambda/n, \Lambda, 0)$ and hence, Corollary Theorem B.0.2 (c) yields $w - w' \equiv t$ for a constant t . \square

Definition 3.1.1. *Let $(M, x) \in \mathcal{S}^n \times \bar{\Omega}$.*

- (i) *For each $\delta \in (0, 1)$, we denote $w^\delta(\cdot; M, x)$ by the unique bounded 1-periodic solution of (3.1.3) and $\tilde{w}^\delta(\cdot; M, x) = w^\delta(\cdot; M, x) - w^\delta(0; M, x)$ in \mathbb{R}^n . By the uniqueness of the solution, we can understand $w^\delta(y; \cdot, \cdot)$ as the mapping $(M, x) \mapsto w^\delta(y; M, x)$ defined on $\mathcal{S}^n \times \bar{\Omega}$ for each $y \in \mathbb{R}^n$.*

- (ii) In a similar way, we write $\bar{F}(M, x)$ by the unique number γ of (3.1.1) and $w(\cdot; M, x)$ by the bounded 1-periodic solution of (3.1.1) normalized by $w(0; M, x) = 0$. Again the uniqueness allows us to understand \bar{F} [resp., $w(y; \cdot, \cdot)$] for each $y \in \mathbb{R}^n$] as the mapping $(M, x) \mapsto \bar{F}(M, x)$ [resp., $w(y; M, x)$] defined on $\mathcal{S}^n \times \bar{\Omega}$.

Note that (3.1.1) now reads

$$(3.1.12) \quad \begin{cases} F(D_y^2 w + M, x, y) = \bar{F}(M, x) & \text{in } \mathbb{R}^n, \\ w \text{ is 1-periodic.} \end{cases}$$

Lemma 3.1.5. For any $L > 0$ and $(M, x), (M', x') \in \bar{B}_L \times \bar{\Omega}$, we have

$$(3.1.13) \quad \left\| \delta w^\delta(\cdot; M', x') - \delta w^\delta(\cdot; M, x) \right\|_{L^\infty(\mathbb{R}^n)} \leq C_L(\|M' - M\| + |x' - x|),$$

and

$$(3.1.14) \quad \left\| \tilde{w}^\delta(\cdot; M', x') - \tilde{w}^\delta(\cdot; M, x) \right\|_{L^\infty(\mathbb{R}^n)} \leq C_L(\|M' - M\| + |x' - x|).$$

Proof. For brevity, let us denote by v_1^δ [resp., v_2^δ] the function $w^\delta(\cdot; M', x')$ [resp., $w^\delta(\cdot; M, x)$]. Also we write by \tilde{v}_1^δ [resp., \tilde{v}_2^δ] the function $\tilde{w}^\delta(\cdot; M', x')$ [resp., $\tilde{w}^\delta(\cdot; M, x)$].

We prove (3.1.13) first. By the Lipschitz continuity of F , we get

$$F(D_y^2 v_2^\delta + M, x, y) \geq \delta v_2^\delta - \tau_L(\|M' - M\| + |x' - x|)$$

which implies that $v_2^\delta - \delta^{-1}\tau_L(\|M' - M\| + |x' - x|)$ is a subsolution of (3.1.3). By the comparison principle (Theorem B.0.1), we arrive at

$$\delta v_2^\delta - \delta v_1^\delta \leq \tau_L(\|M' - M\| + |x' - x|) \quad \text{in } \mathbb{R}^n.$$

By a similar argument, we obtain (3.1.13) with $C_L \geq \tau_L$.

Now we move on to the proof of (3.1.14). The main idea is to use the linearisation of F . Define

$$\begin{aligned} a_{ij}(y) &= \int_0^1 F_{p_{ij}}(t\{D^2 v_1^\delta + M'\} + (1-t)\{D^2 v_2^\delta + M\}, tx' + (1-t)x, y) dt, \\ b_k(y) &= \int_0^1 F_{x_k}(t\{D^2 v_1^\delta + M'\} + (1-t)\{D^2 v_2^\delta + M\}, tx' + (1-t)x, y) dt. \end{aligned}$$

It is immediate from the structure conditions (F1)-(F3) that a_{ij} and b_k ($i, j, k = 1, \dots, n$) are 1-periodic and bounded uniformly in \mathbb{R}^n by the Lipschitz constant of F . We highlight here that it is related only to the Lipschitz regularity of F ; indeed, if F is (only) Lipschitz continuous on $\mathcal{S}^n \times \bar{\Omega} \times \mathbb{R}^n$, then $F_{p_{ij}}$ and F_{x_k} exist a.e. and bounded uniformly by the Lipschitz constant of F . Henceforth, $a_{ij}(y)$ and $b_k(y)$ exist for all $y \in \mathbb{R}^n$ and bounded by the same constant. On the other hand, we also see that (a_{ij}) is uniformly elliptic with the same ellipticity constants λ and Λ of F .

Now $v^\delta := \tilde{v}_1^\delta - \tilde{v}_2^\delta \in C^{2,\alpha}(\mathbb{R}^n)$ solves

$$a_{ij} D_{ij} v^\delta + a_{ij}(M'_{ij} - M_{ij}) + b_k(x'_k - x_k) = \delta v_1^\delta - \delta v_2^\delta \quad \text{in } \mathbb{R}^n.$$

For simplicity, we also denote by ϕ^δ the term $\delta v_1^\delta - \delta v_2^\delta - a_{ij}(M'_{ij} - M_{ij}) - b_k(x'_k - x_k)$.

By (3.1.13) we know that

$$\left\| \phi^\delta \right\|_{L^\infty(\mathbb{R}^n)} \leq 2\tau_L(\|M' - M\| + |x' - x|).$$

Moreover, $v^\delta(0) = \tilde{v}_1^\delta(0) - \tilde{v}_2^\delta(0) = 0$. Since a_{ij} is bounded periodic and uniformly elliptic with ellipticity constants λ and Λ , we can argue in the manner of Lemma 2.1.4. To be more specific, we consider $\widehat{v}^\delta = v^\delta - \min_{\mathbb{R}^n} v^\delta = v^\delta - \min_{Q_1} v^\delta$ and apply the Harnack inequality (Proposition A.0.1 (c)) to get bounded oscillation

$$\operatorname{osc}_{\mathbb{R}^n} v^\delta = \sup_{Q_1} \widehat{v}^\delta \leq c_0 \left\| \phi^\delta \right\|_{L^\infty(Q_2)} \leq 2c_0 \tau_L (\|M' - M\| + |x' - x|).$$

Then since $\tilde{v}^\delta = v^\delta - v^\delta(0) = v^\delta$ in \mathbb{R}^n , we finally obtain

$$\left\| v^\delta \right\|_{\mathbb{R}^n} \leq \operatorname{osc}_{\mathbb{R}^n} v^\delta \leq 2c_0 \tau_L (\|M' - M\| + |x' - x|).$$

The proof is thus complete after choosing $C_L = \max\{\tau_L, 2c_0 \tau_L\}$. \square

Lemma 3.1.6. *The convergence in (3.1.11) is uniform in $(M, x) \in \overline{B}_L \times \overline{\Omega}$ for each $L > 0$; i.e.,*

$$(3.1.15) \quad \lim_{\delta \rightarrow 0} \left\{ \sup_{(M, x) \in \overline{B}_L \times \overline{\Omega}} \left(\left\| \delta w^\delta(\cdot; M, x) - \bar{F}(M, x) \right\|_{L^\infty(\mathbb{R}^n)} + \left\| \tilde{w}^\delta(\cdot; M, x) - w(\cdot; M, x) \right\|_{C^2(\mathbb{R}^n)} \right) \right\} = 0.$$

Proof. Fix $L > 0$. Put $C_L = \sup\{C_{\|M\|} : M \in \overline{B}_L\}$ and then take $\tilde{C}_L = \max\{\sigma(1 + L), C_L\}$. Then it follows from (3.1.5) and (3.1.9) that

$$(3.1.16) \quad \sup_{0 < \delta < 1} \sup_{(M, x) \in \overline{B}_L \times \overline{\Omega}} \max \left\{ \left\| \delta w^\delta(\cdot; M, x) \right\|_{L^\infty(\mathbb{R}^n)}, \left\| \tilde{w}^\delta(\cdot; M, x) \right\|_{C^{2,\alpha}(\mathbb{R}^n)} \right\} \leq \tilde{C}_L.$$

The above uniform estimates allow us to extract a subsequence $\{\delta_k w^{\delta_k}\}_{k=1}^\infty$ [resp. $\{\tilde{w}^{\delta_k}\}_{k=1}^\infty$] from $\{\delta w^\delta\}_{0 < \delta < 1}$ [resp. $\{\tilde{w}^\delta\}_{0 < \delta < 1}$] such that (3.1.11) holds regardless of a particular choice of $(M, x) \in \overline{B}_L \times \overline{\Omega}$. The rest of the proof is the same with that in Lemma 3.1.2. \square

It is an immediate consequence of Lemma 3.1.5 and 3.1.6 that the effective operator \bar{F} and the corresponding corrector $w(y; \cdot, \cdot)$ is locally Lipschitz continuous (uniform in y). Due to its particular role in the rest of this paper, we present the statement without proof.

Lemma 3.1.7. *\bar{F} and $w(y; \cdot, \cdot)$ are Lipschitz continuous locally in \mathcal{S}^n and globally in $\overline{\Omega}$. Moreover, the Lipschitz continuity of the latter is uniform in $y \in \mathbb{R}^n$.*

There are additional properties of \bar{F} . A more general proof is contained in [E2], but we present it for the sake of completeness. In addition, we make a slight adjustment of the proof according to our situation; the main difference is that we have $C^{2,\alpha}$ -corrector, which makes the proof simpler.

Lemma 3.1.8. (i) \bar{F} is uniformly elliptic with the same constants λ and Λ of F .
(ii) \bar{F} is concave on \mathcal{S}^n .

Proof. Let $(M, x) \in \mathcal{S}^n \times \overline{\Omega}$ and $N \in \mathcal{S}^n$ with $N \geq 0$. Think of w^M and w^{M+N} which solve respectively (3.1.12) and

$$(3.1.17) \quad F(D_y^2 w^{M+N} + M + N, x, y) = \bar{F}(M + N, x) \quad \text{in } \mathbb{R}^n.$$

Suppose to the contrary that we could choose N so as to satisfy

$$\bar{F}(M + N, x) - \bar{F}(M, x) < \lambda \|N\|.$$

It is no restriction to assume that $w^{M+N} < w^M$ in \mathbb{R}^n , since adding a constant to w^M and w^{M+N} does not affect (3.1.12). As F being uniformly elliptic with the constants λ and Λ , we have

$$F(D_y^2 w^{M+N} + M, x, y) \leq \bar{F}(M + N, x) - \lambda \|N\| < \bar{F}(M, x).$$

This implies that w^{M+N} is a supersolution to (3.1.12). By comparison, $w^{M+N} \geq w^M$ in \mathbb{R}^n , which contradicts our hypothesis. Thus, it must hold that

$$\bar{F}(M + N, x) - \bar{F}(M, x) \geq \lambda \|N\|$$

for any $M, N \in \mathcal{S}^n$ with $N \geq 0$ and $x \in \bar{\Omega}$. In the same way we are able to prove that the left hand side of the last inequality is bounded above by $\Lambda \|N\|$. This completes the proof of (i).

Now we establish the proof of (ii). Let $M, N \in \mathcal{S}^n$ and $x \in \bar{\Omega}$ be given. It suffices to prove that

$$(3.1.18) \quad \bar{F}\left(\frac{M+N}{2}, x\right) \geq \frac{\bar{F}(M, x) + \bar{F}(N, x)}{2},$$

because of the continuity of \bar{F} (which was proven in (ii)). Denote w^M , w^N and $w^{(M+N)/2}$ by the solutions of (3.1.12) respectively with M , N and

$$(3.1.19) \quad F\left(D_y^2 w^{(M+N)/2} + \frac{M+N}{2}, x, y\right) = \bar{F}\left(\frac{M+N}{2}, x\right) \quad \text{in } \mathbb{R}^n.$$

Adding a constant to $w^{(M+N)/2}$ if needs be, we may assume that $w^{(M+N)/2} < \frac{w^M + w^N}{2}$ in \mathbb{R}^n . Suppose to the contrary that (3.1.18) fails to hold for some M, N . Then we obtain from the concavity of F that

$$\bar{F}\left(\frac{M+N}{2}, x\right) < \frac{\bar{F}(M, x) + \bar{F}(N, x)}{2} \leq F\left(\frac{M+N}{2} + D_y^2 \left(\frac{w^M + w^N}{2}\right), x, y\right)$$

in \mathbb{R}^n . By a comparison with (3.1.19), we get $w^{(M+N)/2} \geq \frac{w^M + w^N}{2}$ in \mathbb{R}^n , which is a contradiction. It finishes the proof of (iii). \square

As we mentioned in the beginning of this section, we determine the equation which u solves in the viscosity sense.

Lemma 3.1.9. *Assume that $F \in C(\mathcal{S}^n \times \bar{\Omega} \times \mathbb{R}^n)$ satisfy the hypotheses (F1)-(F4). Then the function u from Lemma 3.1.1 solves*

$$(\bar{F}) \quad \begin{cases} \bar{F}(D^2 u, x) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Moreover, u is unique and belongs to the class of $C^{2,\alpha}(\bar{\Omega})$.

Proof. The proof of that u is a viscosity solution of (\bar{F}) is similar to that of Lemma 2.1.6. Instead of using strong maximum principle, one may take advantage of Theorem B.0.1 (a). The details are left to the readers.

As long as we know that u solves (\bar{F}) , the fact that $u \in C^{2,\alpha}(\Omega)$ follows readily from Theorem B.0.2 (e). The proof is similar to that in Lemma 3.1.3, so the details are omitted; instead of taking advantage of (F1)-(F4), we use Lemma 3.1.8 (i)-(iii). We make a remark here that the exponent α is the same with which we chose in

Lemma 3.1.3 because the ellipticity constants of \bar{F} coincide with those of F (Lemma 3.1.8 (i)). \square

3.2. Regularity of the effective operator and the corrector. In the previous subsection, we observed that the Lipschitz regularity of F , in particular in the (M, x) -variable, yields the Lipschitz regularity of \bar{F} and $w(y; \cdot, \cdot)$, where the latter is uniform in $y \in \mathbb{R}^n$. It is then natural to ask whether a higher regularity of F in (M, x) -variable gives a higher regularity for \bar{F} and $w(y; \cdot, \cdot)$, and we prove in this subsection that the answer is affirmative. Specifically, we observe that they have the same regularity as F does. This regularity result plays the key role in the rest of this paper, especially in seeking higher order interior correctors. To be precise, we observe the following.

Proposition 3.2.1. \bar{F} and $w(y; \cdot, \cdot)$ are $C^{m,1}$ locally in \mathcal{S}^n and globally in $\bar{\Omega}$ and for any $L > 0$,

$$(3.2.1) \quad \|\bar{F}\|_{C^{m,1}(\bar{B}_L \times \bar{\Omega})} + \|w(y; \cdot, \cdot)\|_{C^{m,1}(\bar{B}_L \times \bar{\Omega})} \leq C_{L,m}$$

Moreover, for any $(M', x'), (M, x) \in \bar{B}_L \times \bar{\Omega}$ there holds

$$(3.2.2) \quad \sum_{0 \leq i+j \leq m-1} \|D_p^i D_x^j w(\cdot; M', x') - D_p^i D_x^j w(\cdot; M, x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{L,m}(\|M' - M\| + |x' - x|).$$

Remark. Note that (3.2.2) implies that $D_p^i w(y; \cdot, \cdot) \in C^{m-1,1}(\bar{B}_L \times \bar{\Omega})$ for $i = 1, 2$. This will turn out as the coupling effect which we addressed in Introduction.

Before we begin the proof, let us illustrate the heuristics of our argument. In the first place, we only assume that F satisfies the structure condition (F3) with $m = 1$, which means that F is $C^{1,1}$ locally in \mathcal{S}^n and but globally in $\bar{\Omega} \times \mathbb{R}^n$, and end up with the conclusion that \bar{F} and $w(y; \cdot, \cdot)$ are also $C^{1,1}$ locally in \mathcal{S}^n and globally in $\bar{\Omega}$. We also observe that the equation, which involves the partial derivatives of \bar{F} and $w(y; \cdot, \cdot)$ in M and x -variable solves, satisfies the same structure conditions of F . This implies that under our original assumption (F3) we are able to iterate the argument to get $C^{m,1}$ regularity of \bar{F} and $w(y; \cdot, \cdot)$ which is local in \mathcal{S}^n and global in $\bar{\Omega}$. For this reason, we discuss the proof of the former observation in detail and skip the redundancy in the induction argument.

The rest of this subsection is devoted to the proof of the first part, namely the $C^{1,1}$ regularity of \bar{F} and $w(y; \cdot, \cdot)$. As the first step, we consider the linearized equation (3.2.5), which $w^\delta(\cdot; M', x') - w^\delta(\cdot; M, x)$ satisfies, and prove that the coefficients are uniformly elliptic with the same ellipticity constants of F and are $C^{0,\alpha}(\mathbb{R}^n)$ whose $C^{0,\alpha}(\mathbb{R}^n)$ -norm is independent of δ . This enables us to apply the interior Schauder estimate to the linearized equation, and henceforth, replace the L^∞ -norm in (3.1.13) and (3.1.14) by $C^{2,\alpha}$ -norm.

Lemma 3.2.1. For each $L > 0$ and $(M, x), (M', x') \in \bar{B}_L \times \bar{\Omega}$, there hold

$$(3.2.3) \quad \sup_{0 < \delta < 1} \|\delta w^\delta(\cdot; M', x') - \delta w^\delta(\cdot; M, x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_L(\|M' - M\| + |x' - x|)$$

and

$$(3.2.4) \quad \sup_{0 < \delta < 1} \|\tilde{w}^\delta(\cdot; M', x') - \tilde{w}^\delta(\cdot; M, x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_L(\|M' - M\| + |x' - x|),$$

Proof. The main idea has been already introduced in the proof of Lemma 3.1.5. For simplicity we write $v_1^\delta = w^\delta(\cdot; M', x')$ and $v_2^\delta = w^\delta(\cdot; M, x)$ and $\tilde{v}_1^\delta, \tilde{v}_2^\delta$ correspondingly. Also $v^\delta = v_1^\delta - v_2^\delta$ and $\tilde{v}^\delta = \tilde{v}_1^\delta - \tilde{v}_2^\delta$. Then both v^δ and \tilde{v}^δ satisfy the following linearized equation

$$(3.2.5) \quad a_{ij}^\delta D_{ij} w + a_{ij}^\delta (M'_{ij} - M_{ij}) + b_k^\delta (x'_k - x_k) = \delta v^\delta \quad \text{in } \mathbb{R}^n,$$

where

$$\begin{aligned} a_{ij}^\delta(y) &= \int_0^1 F_{p_{ij}}(t\{D^2 v_1^\delta + M'\} + (1-t)\{D^2 v_2^\delta + M\}, tx' + (1-t)x, y) dt, \\ b_k^\delta(y) &= \int_0^1 F_{x_k}(t\{D^2 v_1^\delta + M'\} + (1-t)\{D^2 v_2^\delta + M\}, tx' + (1-t)x, y) dt. \end{aligned}$$

It is clear that a_{ij}^δ and b_k^δ are 1-periodic functions. The coefficients a_{ij}^δ are also uniformly elliptic with the same constants λ and Λ in the sense of (L1), which can be easily derived from (F1) and the continuous differentiability of F . Note as well that the ellipticity constants have no dependence on δ .

Now by the $C^{1,1}$ assumption on F , we obtain a uniform $C^{0,\alpha}(\mathbb{R}^n)$ -estimate on a_{ij}^δ and b_k^δ , which is the major difference from the situation in Lemma 3.1.5. Since the proof for b_k^δ is the same, we only provide the proof for a_{ij}^δ . By (3.1.10) that for any $t \in [0, 1]$,

$$\begin{aligned} \left\| t\{D^2 v_1^\delta + M'\} + (1-t)\{D^2 v_2^\delta + M\} \right\|_{L^\infty(\mathbb{R}^n)} &\leq t(C_{\|M'\|} + \|M'\|) + (1-t)(C_{\|M\|} + \|M\|) \\ &\leq C_L + L. \end{aligned}$$

Hence, we deduce from the condition (F3) that

$$\left\| F_{p_{ij}}(t\{D^2 v_1^\delta + M'\} + (1-t)\{D^2 v_2^\delta + M\}, tx' + (1-t)x, \cdot) \right\|_{L^\infty(\mathbb{R}^n)} \leq \tau_{C_L+L}.$$

Thus,

$$\left\| a_{ij}^\delta \right\|_{L^\infty(\mathbb{R}^n)} \leq \tau_{C_L+L} \leq \sigma(C_L + L + 1).$$

To estimate the $C^{0,\alpha}$ -seminorm of a_{ij}^δ , we need the condition (F3) for $m = 1$, i.e., the $C^{1,1}$ regularity of F . It is enough for us to achieve the uniform estimate inside a periodic cube. Hence, we compute the difference of $F_{p_{ij}}(N_t^\delta, x_t, \cdot)$ at two distinct points $y_1, y_2 \in Q_1$, where $N_t^\delta = t\{D^2 v_1^\delta + M'\} + (1-t)\{D^2 v_2^\delta + M\}$ and $x_t = tx + (1-t)x'$. Again by (3.1.10),

$$\begin{aligned} |F_{p_{ij}}(N_t^\delta(y_1), x_t, y_1) - F_{p_{ij}}(N_t^\delta(y_2), x_t, y_2)| &\leq \tau_{C_L+L} (\|N_t^\delta(y_1) - N_t^\delta(y_2)\| + |y_1 - y_2|) \\ &\leq \tau_{C_L+L} (C_L |y_1 - y_2|^\alpha + |y_1 - y_2|) \\ &\leq \tilde{C}_L |y_1 - y_2|^\alpha, \end{aligned}$$

with $\tilde{C}_L = \sigma(C_L + L + 1)(C_L + 1)$. Thus, the periodicity of a_{ij}^δ yields

$$[a_{ij}^\delta]_{C^{0,\alpha}(Q_1)} \leq \tilde{C}_L.$$

Note that the periodicity of a_{ij}^δ indeed implies that $[a_{ij}^\delta]_{C^{0,\alpha}(\mathbb{R}^n)} \leq [a_{ij}^\delta]_{C^{0,\alpha}(Q_1)}$. This combined with the above L^∞ -bound on a_{ij}^δ proves that

$$\left\| a_{ij}^\delta \right\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq 2\tilde{C}_L.$$

Recall also from (3.1.7) that

$$\|\delta v^\delta\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq \|\delta v_1^\delta\|_{C^{0,\alpha}(\mathbb{R}^n)} + \|\delta v_2^\delta\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq 6c_0\sigma(1+L).$$

Therefore, we may apply the interior Schauder estimate to (3.2.5) in a ball $B_{\sqrt{n}}$ containing a periodic cube to get the conclusion. For details, see the proofs of Lemma 2.1.3 and 2.1.4. \square

As a corollary, we obtain the same Lipschitz continuity of $w(y; \cdot, \cdot)$ in (M, x) -variable which is uniform in the $C^{2,\alpha}(\mathbb{R}^n)$ -norm.

Lemma 3.2.2. *For each $L > 0$ and $(M, x), (M', x') \in \bar{B}_L \times \bar{\Omega}$, there holds*

$$(3.2.6) \quad \|w(\cdot; M', x') - w(\cdot; M, x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_L(\|M' - M\| + |x' - x|).$$

Proof. Apply the uniform convergence (Lemma 3.1.6) to get

$$\|w(\cdot; M', x') - w(\cdot; M, x)\|_{C^2(\mathbb{R}^n)} \leq C_L(\|M' - M\| + |x' - x|).$$

Then use the uniform boundedness of $C^{2,\alpha}(\mathbb{R}^n)$ -norm of $w(\cdot; M', x') - w(\cdot; M, x)$ (Lemma 3.1.2) and the compactness argument (Theorem C.0.3) to improve this inequality to $C^{2,\alpha}(\mathbb{R}^n)$ -norm. \square

Next we linearize the equation (3.1.12), which gives the equation that $(w(\cdot; M + hE^{kl}, x) - w(\cdot; M, x))/h$ [resp., $(w(\cdot; M, x + hE_k) - w(\cdot; M, x))/h$] satisfies. The same argument as in Lemma 3.2.1 would apply to the coefficients of the resulting equation so that their $C^{0,\alpha}(\mathbb{R}^n)$ -norms are bounded and independent of h . This implies the stability of the linearized equation under the limit $h \rightarrow 0$. Moreover, we observe that the linearized equation is in the same class of (2.1.2). It allows us to argue as before and to obtain the limit solution and the unique limit constant, which are exactly the partial derivative of $w(y; \cdot, \cdot)$ and \bar{F} in M [resp., x]-variable.

Let us make our argument precise.

Lemma 3.2.3. *There exist $\bar{F}_{p_{kl}}, \bar{F}_{x_k}, D_{p_{kl}}w(y; \cdot, \cdot)$ and $D_{x_k}w(y; \cdot, \cdot)$ for each $y \in \mathbb{R}^n$ on $S^n \times \bar{\Omega}$. In addition, there hold for any $L > 0$ and $(M, x) \in \bar{B}_L \times \bar{\Omega}$,*

$$(3.2.7) \quad |\bar{F}_{p_{kl}}(M, x)|, |\bar{F}_{x_k}(M, x)|, \|D_{p_{kl}}w(\cdot; M, x)\|_{C^{2,\alpha}(\mathbb{R}^n)}, \|D_{x_k}w(\cdot; M, x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_L.$$

Proof. Here we only provide the proof for the M -partial derivatives of \bar{F} and $w(y; \cdot, \cdot)$. The argument for the x -partial derivatives is similar so we omit it to avoid the redundancy.

Pick any $L > 0$ and $(M, x) \in \bar{B}_L \times \bar{\Omega}$. By v_h we denote $h^{-1}[w(\cdot; M + hE^{kl}, x) - w(\cdot; M, x)]$. As we linearize the equation (3.1.12) with $M + hE^{kl}$ and M , and divide the both sides by h , we observe that v_h satisfies

$$(3.2.8) \quad a_{ij,h}D_{ij}v_h + a_{kl,h} = \gamma_h$$

where

$$\begin{aligned} a_{ij,h} &= \int_0^1 F_{p_{ij}}(tD_y^2w(\cdot; M + hE^{kl}, x) + (1-t)D_y^2w(\cdot; M, x) + M + thE^{kl}, x, \cdot) dt, \\ \gamma_h &= \frac{\bar{F}(M + hE^{kl}, x) - \bar{F}(M, x)}{h}. \end{aligned}$$

By following the argument in the proof of Lemma 3.2.1, we observe that for any h with $|h|$ small, $a_{ij,h}$ is also uniformly elliptic with the ellipticity constants λ and Λ (in the sense of (L1)), and belongs to $C^{0,\alpha}(\mathbb{R}^n)$ with

$$\|a_{ij,h}\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq c_L.$$

Also we know from Lemma 3.1.7 that

$$|\gamma_h| \leq \tilde{c}_L.$$

Therefore, the linearized equation (3.2.8) belongs to the same class of (2.1.2) in the sense that the variable coefficients are uniformly elliptic and $C^{0,\alpha}$ and the right hand side is bounded uniformly with respect to the parameter, which in the former case is h while δ in the latter case. (Note that as long as we know the existence of the solution w^δ of (2.1.2), we may treat δw^δ as an external forcing term.)

One may notice that the variable coefficients of (2.1.2) are fixed with respect to the parameter, while those of (3.2.8) vary. However, the proof of Lemma 2.1.2 is still applicable in this case because we have a uniform convergence of $a_{ij,h}$ as $h \rightarrow 0$. Indeed, Lemma 3.2.2 implies that

$$a_{ij,h} \longrightarrow a_{ij} = F_{p_{ij}}(D_y^2 w(\cdot; M, x) + M, x, \cdot) \quad \text{uniformly in } \mathbb{R}^n,$$

as $h \rightarrow 0$.

Consequently, we deduce from the proof of Lemma 2.1.2 that there exist a unique constant γ and a bounded 1-periodic function $v \in C^{2,\alpha}(\mathbb{R}^n)$ such that

$$|\gamma_h - \gamma| + \|v_h - v\|_{C^2(\mathbb{R}^n)} \longrightarrow 0$$

as $h \rightarrow 0$ and that v satisfies

$$(3.2.9) \quad a_{ij} D_{ij} v + a_{kl} = \gamma \quad \text{in } \mathbb{R}^n.$$

By definition, $\gamma = \bar{F}_{p_{kl}}(M, x)$ and $v = D_{p_{kl}} w(\cdot; M, x)$. One should notice that we do not force $v(0)$ to be 0 here; otherwise, we could not say that $v = D_{p_{kl}} w(\cdot; M, x)$. The uniform estimate (3.2.7) now follows from Lemma 3.1.7 and 3.2.2. \square

Now we are left with proving that the partial derivatives of \bar{F} and $w(y; \cdot, \cdot)$ are Lipschitz continuous locally in \mathcal{S}^n and globally in $\bar{\Omega}$. The idea is to apply Lemma 3.1.7 to the partial derivatives. To do so, we need to justify that the equation, which the partial derivatives of \bar{F} and $w(y; \cdot, \cdot)$ satisfy, can be approximated by δ -penalization problems just as (3.1.3) approximates (3.1.1), and that those δ -penalization problems are in the same class of (3.1.3).

Lemma 3.2.4. $\bar{F}_{p_{kl}}, \bar{F}_{x_k}, D_{p_{kl}} w(y; \cdot, \cdot)$ and $D_{x_k} w(y; \cdot, \cdot)$ are Lipschitz continuous locally in \mathcal{S}^n and globally in $\bar{\Omega}$. Moreover, the Lipschitz continuity of the latter two is uniform $y \in \mathbb{R}^n$.

Proof. Here we only present the proof for the M -partial derivatives. The proof for the x -partial derivatives is the same, and we leave it to the readers.

To make our discussion clear and concrete, let us consider (3.2.9), which $\gamma = \bar{F}_{p_{kl}}$ and $v = D_{p_{kl}} w(\cdot; M, x)$ satisfy. In the proof of Lemma 3.2.3, we derived (3.2.9) from (3.2.8) by, roughly speaking, sending h to 0. However, it fails to be our desired δ -penalization problem, since we would not be successful in obtaining the Lipschitz continuity of $\bar{F}_{p_{kl}}$ via this penalization.

To get a right penalization we need to go back to beginning and start from the very general linearization problem (3.2.5). Substituting M' [resp., x'] with $M + hE^{kl}$ [resp., x] in this equation one obtains

$$(3.2.10) \quad a_{ij,h}^\delta D_{ij} v_h^\delta + a_{kl,h}^\delta - \delta v_h^\delta = 0 \quad \text{in } \mathbb{R}^n,$$

where

$$\begin{aligned} a_{ij,h}^\delta &= \int_0^1 F_{p_{ij}}(tD_y^2 w^\delta(\cdot; M + hE^{kl}, x) + (1-t)D_y^2 w^\delta(\cdot; M, x) + M + thE^{kl}, x, \cdot) dt, \\ v_h^\delta &= \frac{w^\delta(\cdot; M + hE^{kl}, x) - w^\delta(\cdot; M, x)}{h}. \end{aligned}$$

Then Lemma 3.2.1 provides two observations. Firstly, the $C^{2,\alpha}$ -norm of v_h^δ is uniformly bounded by C_L . Then the Arzela-Ascoli theorem (Theorem C.0.3) ensures the existence of a limit function v^δ , which is bounded 1-periodic and belongs to $C^{2,\alpha}(\mathbb{R}^n)$, for each δ along a subsequence of h . Secondly, $a_{ij,h}^\delta$ converges uniformly in \mathbb{R}^n as $h \rightarrow 0$ to

$$a_{ij}^\delta = F_{p_{ij}}(D_y^2 w^\delta(\cdot; M, x) + M, x, \cdot).$$

Since a_{ij}^δ is also uniformly elliptic with the same ellipticity constants λ and Λ , the closed-ness of the (viscosity) solutions (c.f. the proof of Lemma 2.1.2) implies that the limit function v^δ is indeed a solution of

$$(3.2.11) \quad a_{ij}^\delta D_{ij} v^\delta + a_{kl}^\delta - \delta v^\delta = 0 \quad \text{in } \mathbb{R}^n.$$

Due to the uniqueness of the solution of (3.2.11) (c.f. Lemma 2.1.3), we now know that the limit of $v_h^\delta \rightarrow v^\delta$ (in $C^2(\mathbb{R}^n)$ -norm) takes place for the full sequence of h .

We claim that (3.2.11) is an appropriate penalization of (3.2.9). The dependency of a_{ij}^δ [resp., v^δ] on (M, x) now becomes important, so let us denote a_{ij}^δ [resp., v^δ] by $a_{ij}^\delta(\cdot; M, x)$ [resp., $v^\delta(\cdot; M, x)$] to specify it.

To prove this claim, we are only required to observe the following two facts: first, $a_{ij}^\delta(y; M, x)$ converges uniformly for $y \in \mathbb{R}^n$ and $(M, x) \in \overline{B}_L \times \overline{\Omega}$ for each $L > 0$ to $a_{ij}(y; M, x) = F_{p_{ij}}(D_y^2 w(y; M, x) + M, x, y)$; second, $a_{ij}^\delta(y; \cdot, \cdot)$ is Lipschitz continuous in $\overline{B}_L \times \overline{\Omega}$ uniformly in $y \in \mathbb{R}^n$. The rest of the proof just follows the argument in Lemma 3.1.2 and 3.1.5.

The former is already known. Indeed if we combine Lemma 3.1.6 and 3.2.1, we get

$$\lim_{(\delta, h) \rightarrow (0+, 0)} \sup_{(M, x) \in \overline{B}_L \times \overline{\Omega}} \|a_{ij,h}^\delta(\cdot; M, x) - a_{ij}^\delta(\cdot; M, x)\|_{L^\infty(\mathbb{R}^n)} = 0.$$

This convergence tells us that as we send δ to zero, we come up with an equation

$$a_{ij} D_{ij} v' + a_{kl} = \gamma' \quad \text{in } \mathbb{R}^n,$$

where $(\delta v^\delta, \delta v^\delta) \rightarrow (v', \gamma')$. If we compare this equation to (3.2.9) we are forced to conclude that $\gamma' = \gamma$ and $v' - v \equiv t$ in \mathbb{R}^n , where t is a constant. Otherwise, it would make a contradiction just as in the proof of Lemma 3.1.2.

Thus, we are only left with the latter, i.e., to prove that $a_{ij}^\delta(y; \cdot, \cdot)$ is Lipschitz continuous locally in \mathbb{S}^n and globally in $\overline{\Omega}$, and the continuity is uniform in $y \in \mathbb{R}^n$. To see this, choose any $L > 0$ and $(N, z), (N', z') \in \overline{B}_L \times \overline{\Omega}$. According to (3.1.10), the $C^{2,\alpha}(\mathbb{R}^n)$ -norm of both $w^\delta(\cdot; N, z)$ and $w^\delta(\cdot; N', z')$ is uniformly bounded by C_L . Thus,

the structure condition (F3) together with (3.2.3) yields that there holds uniformly for $y \in \mathbb{R}^n$,

$$\begin{aligned} |a_{ij}^\delta(y; N, z) - a_{ij}^\delta(y; N', z')| &\leq \tau_{C_L}(\|D_y^2 w^\delta(y; N, z) - D_y^2 w^\delta(y; N', z')\| + |z - z'|) \\ &\leq \tilde{C}_L(\|N - N'\| + |z - z'|). \end{aligned}$$

It then allows us to go through the argument in Lemma 3.1.5, which finally completes the proof. \square

We are now in position to present the proof of our main proposition of this subsection.

Proof of Proposition 3.2.1. Observe from Lemma 3.2.4 the first order partial derivatives of \bar{F} and $w(y; \cdot, \cdot)$ satisfies the equations (e.g., (3.2.9)) which belong to the same class of (3.1.1), and admit the δ -approximating problems (e.g., (3.2.11)) which correspond to (3.1.3). Thus, we can repeat the argument used through Lemma 3.2.1-3.2.4 again to get the Lipschitz continuity of the second order partial derivatives of \bar{F} and $w(y; \cdot, \cdot)$. We iterate this process by m -times to reach the conclusion. We leave the details to the readers. \square

3.3. Interior and boundary layer correctors. Now we are in position to construct higher order correctors which correct the error occurring inside of interior and on boundary layer of our physical domain Ω . This subsection involves many iterative arguments, so before we make our argument rigorous, we want to provide the key idea to help the readers' understanding.

First and foremost, we highlight the fact that the asymptotic expansion of u^ε , say

$$u^\varepsilon(x) \simeq u(x) + \varepsilon w_1(x) + \varepsilon^2 w_2(\varepsilon^{-1}x, x) + \cdots + \varepsilon^r w_r(\varepsilon^{-1}x, x),$$

occurs inside of the operator F , which makes a big difference from the linear case; i.e.,

$$\begin{aligned} F(D^2 u^\varepsilon) &\simeq F(D^2 \{u + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots + \varepsilon^r w_r\}) \\ &= F(D_x^2 u + D_y^2 w_2 + \varepsilon[D_x^2 w_1 + D_{x,y} w_2 + D_y^2 w_3] + \cdots \\ &\quad + \varepsilon^{r-2}[D_x^2 w_{r-2} + D_{x,y} w_{r-1} + D_y^2 w_r] + \varepsilon^{r-1}[D_x^2 w_{r-1} + D_{x,y} w_r] + \varepsilon^r D_x^2 w_r). \end{aligned}$$

Here we have dropped the dependency on $(\varepsilon^{-1}x, x)$ and denoted by $D_{x,y}$ the operator $D_x D_y + D_y D_x$. To further simplify our notation, let us introduce $X^k \in \mathcal{S}^n$, $1 \leq k \leq r$, defined by

$$(3.3.1) \quad X^k = \begin{cases} D_x^2 u(\cdot) + D_y^2 w_2(\cdot/\varepsilon, \cdot) & \text{if } k = 0 \\ D_x^2 w_k(\cdot/\varepsilon, \cdot) + D_{x,y} w_{k+1}(\cdot/\varepsilon, \cdot) + D_y^2 w_{k+2}(\cdot/\varepsilon, \cdot) & \text{if } 1 \leq k \leq r-2, \\ D_x^2 w_{r-1}(\cdot/\varepsilon, \cdot) + D_{x,y} w_r(\cdot/\varepsilon, \cdot) & \text{if } k = r-1, \\ D_x^2 w_r(\cdot/\varepsilon, \cdot) & \text{if } k = r, \end{cases}$$

and Y^r defined by

$$(3.3.2) \quad Y^r = X^1 + \varepsilon X^2 + \cdots + \varepsilon^{r-1} X^r.$$

Then a Taylor expansion of F with respect to the Hessian gives,

$$F(X^0 + \varepsilon Y^r) = F(X^0) + \varepsilon F_{p_{ij}}(X^0) Y_{ij}^r + \cdots + \frac{\varepsilon^r}{r!} F_{p_{i_1 j_1} \cdots p_{i_r j_r}}(X^0) Y_{i_1 j_1}^r \cdots Y_{i_r j_r}^r + O(\varepsilon^{r+1}),$$

which would be valid provided that $\|Y^r\|_{L^\infty(\Omega)} \leq C$ with a positive constant independent of ε . This in turn requires us to have a uniform control (i.e., independent of ε) on the supremum norm of second order derivatives of w_k in both x and y -variables.

Moreover, one should note that $Y^r = \sum_{k=1}^r \varepsilon^{k-1} X^k$ is a summation of the terms of different ε -order. For this reason we rearrange the terms in the Taylor expansion according to the ε -power as below.

(3.3.3)

$$\begin{aligned} F(X^0 + \varepsilon Y^r) &= F(X^0) + \varepsilon F_{p_{ij}}(X^0) X_{ij}^1 + \cdots + \varepsilon^r \sum_{l=1}^r \frac{1}{l!} \sum_{n_1 + \cdots + n_l = r} F_{p_{i_1 j_1} \cdots p_{i_l j_l}}(X^0) X_{i_1 j_1}^{n_1} \cdots X_{i_l j_l}^{n_l} \\ &\quad + \sum_{l=1}^r \sum_{r+1 \leq n_1 + \cdots + n_l \leq rl} \frac{\varepsilon^{n_1 + \cdots + n_l}}{l!} F_{p_{i_1 j_1} \cdots p_{i_l j_l}}(X^0) X_{i_1 j_1}^{n_1} \cdots X_{i_l j_l}^{n_l} + O(\varepsilon^{r+1}). \end{aligned}$$

It suggests that we need to find w_1, \dots, w_r in such a way that $F(X^0) = 0$, $F_{p_{ij}}(X^0) X_{ij}^1 = 0$, and so on.

To satisfy $F(X^0) = 0$, w_2 must be chosen such that $D_y^2 w_2 = D_y^2 w(\cdot; D_x^2 u, x)$. Then $F(X^0) = \bar{F}(D_x^2 u) = 0$ by Lemma 3.1.9. Furthermore, one should obtain, for $k = 1, \dots, r-2$,

$$\begin{aligned} 0 &= \sum_{l=1}^k \frac{1}{l!} \sum_{n_1 + \cdots + n_l = k} F_{p_{i_1 j_1} \cdots p_{i_l j_l}}(X^0) X_{i_1 j_1}^{n_1} \cdots X_{i_l j_l}^{n_l} \\ (3.3.4) \quad &= F_{p_{ij}}(X^0) X_{ij}^k + \sum_{l=2}^k \frac{1}{l!} \sum_{n_1 + \cdots + n_l = k} F_{p_{i_1 j_1} \cdots p_{i_l j_l}}(X^0) X_{i_1 j_1}^{n_1} \cdots X_{i_l j_l}^{n_l} \\ &= F_{p_{ij}}(X^0) D_{y_i y_j} w_{k+2} + \Phi_{k+2}, \end{aligned}$$

which yields the equation for w_k , where

$$\Phi_{k+2} = F_{p_{ij}}(X^0) D_{x_i x_j} w_k + 2F_{p_{ij}}(X^0) D_{x_i y_j} w_{k+1} + \sum_{l=2}^k \frac{1}{l!} \sum_{n_1 + \cdots + n_l = k} F_{p_{i_1 j_1} \cdots p_{i_l j_l}}(X^0) X_{i_1 j_1}^{n_1} \cdots X_{i_l j_l}^{n_l}.$$

One may notice that the rightmost term of Φ_k only involves X^l , l up to $k-1$, which means that the correctors of order greater than or equal to $k+2$ do not appear in Φ_k . Thus, we may find w_{k+2} by an iterative argument. Moreover, since w_{k+2} makes the ε^k -th order term in (3.3.3) to vanish, there is no opportunity to kill the ε^{r-1} and ε^r -th order terms; the same happened in the linear setting. This in turn suggests that we can have at most

$$F(X^0 + \varepsilon Y^r) = O(\varepsilon^{r-1}),$$

which would lead us to $O(\varepsilon^{r-1})$ -rate of convergence (Theorem 1.2.1). Finally we make a remark that as in the linear case, we would come up with the compatibility condition of w_{k+2} , which determines uniquely w_k . Unlike the linear case (Lemma 2.2.1), however, this relationship is more hidden in the induction argument. We will discuss this issue in the proof in more detail.

Now we make our argument rigorous. Throughout this subsection we set $m \geq 2$. First we enhance the regularity of u , since now we have $\bar{F} \in C^{m,1}$.

Lemma 3.3.1. *Assume that F verifies the hypotheses (F1)-(F4). Then $u \in C^{m+2,\alpha}(\overline{\Omega})$ and*

$$\|u\|_{C^{m+2,\alpha}(\overline{\Omega})} \leq C_{m,g,\Omega}.$$

Proof. By Proposition 3.2.1 we know that F is $C^{1,1}$ locally in \mathcal{S}^n and globally in $\overline{\Omega}$. Since u solves (\overline{F}) where $g \in C^{m+2,1}(\overline{\Omega})$ and $\partial\Omega \in C^{m+2,1}$, the regularity theory (Theorem B.0.2 (f)) implies that $u \in C^{m+2,\alpha}(\overline{\Omega})$ and

$$\|u\|_{C^{m+2,\alpha}(\overline{\Omega})} \leq C_{\overline{F},\Omega}(\|u\|_{L^\infty(\Omega)} + \|g\|_{C^{m+2,1}(\overline{\Omega})}),$$

where $C_{\overline{F},\Omega}$ is a constant depending only on the derivatives of \overline{F} up to m -th order, and on Ω . By (3.2.1), $C_{\overline{F},\Omega}$ in turn depends only on the constants appearing in the structure conditions (F1)-(F4) and m . By an a priori estimate, on the other hand, we may bound the supremum norm of u by a constant depending only on λ, Λ, Ω and $\|g\|_{L^\infty(\Omega)}$. It completes the proof. \square

Next we construct the interior higher order correctors. The regularity theory established in Subsection 3.2 now plays an essential role in proving the existence of the correctors and obtaining a uniform control on L^∞ -bound of their second order derivatives.

Lemma 3.3.2. *Suppose $m \geq 2$. Then there exist a family of non-trivial 1-periodic functions $\{w_k : \mathbb{R}^n \times \overline{\Omega} \rightarrow \mathbb{R}\}_{1 \leq k \leq [\frac{m}{2}]+1}$ for which the following holds.*

(i) $w_k(\cdot, x) \in C^{2,\alpha}(\mathbb{R}^n)$ uniformly for all $x \in \overline{\Omega}$ and

$$(3.3.5) \quad \|w_k(\cdot, x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{m,k,g,\Omega}.$$

(ii) $w_k(y, \cdot) \in C^{m-2k+2,1}(\overline{\Omega})$ uniformly for all $y \in \mathbb{R}^n$ and

$$(3.3.6) \quad \|w_k(y, \cdot)\|_{C^{m-2k+2,1}(\overline{\Omega})} \leq C_{m,k,g,\Omega}.$$

Moreover, there holds for any $x_1, x_2 \in \overline{\Omega}$ that

$$(3.3.7) \quad \sum_{l=0}^{m-2k+1} \|D_x^l w_k(\cdot, x_1) - D_x^l w_k(\cdot, x_2)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{m,k,g,\Omega} |x_1 - x_2|.$$

(iii) Provided that $k \geq 3$, for each $x \in \overline{\Omega}$, $w_k(\cdot, x)$ solves

$$(3.3.8) \quad a_{ij}(\cdot, x) D_{y_i y_j} w_k(\cdot, x) + \Phi_k(\cdot, x) = 0 \quad \text{in } \mathbb{R}^n,$$

where

$$\begin{aligned} \Phi_k &= a_{ij} D_{x_i x_j} w_{k-2} + 2a_{ij} D_{x_i y_j} w_{k-1} + \sum_{l=2}^{k-2} \frac{1}{l!} \sum_{n_1+\dots+n_l=k-2} a_{i_1 j_1 \dots i_l j_l} X_{i_1 j_1}^{n_1} \dots X_{i_l j_l}^{n_l}, \\ X_{i_r j_r}^{n_r} &= D_{x_i x_j} w_{n_r} + 2D_{x_i y_j} w_{n_r+1} + D_{y_i y_j} w_{n_r+2}, \quad r = 1, \dots, l, \\ a_{i_1 j_1 \dots i_l j_l} &= F_{p_{i_1 j_1} \dots p_{i_l j_l}}(D_x^2 u + D_y^2 w(\cdot; D_x^2 u, \cdot), \cdot, \cdot), \quad l = 1, \dots, k-2. \end{aligned}$$

Proof. We are going to use an induction argument to construct families of functions $\{\psi_k : \overline{\Omega} \rightarrow \mathbb{R}\}_{-1 \leq k \leq [\frac{m}{2}]+1}$ and $\{\phi_k : \mathbb{R}^n \times \overline{\Omega} \rightarrow \mathbb{R}\}_{1 \leq k \leq [\frac{m}{2}]+1}$ which verify the following conditions:

(IP1) $\phi_k(\cdot, x) \in C^{2,\alpha}(\mathbb{R}^n)$ uniformly for all $x \in \overline{\Omega}$ and

$$\|\phi_k(\cdot, x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{m,k,g,\Omega}.$$

(IP2) $\phi_k(y, \cdot) \in C^{m-2k+4,1}(\overline{\Omega})$ uniformly for $y \in \mathbb{R}^n$ and

$$\|\phi_k(y, \cdot)\|_{C^{m-2k+4,1}(\overline{\Omega})} \leq \tilde{C}_{m,k,g,\Omega}.$$

Moreover, $\phi_k(0, \cdot) = 0$ in $\overline{\Omega}$ and there holds for any $x_1, x_2 \in \overline{\Omega}$ that

$$\sum_{l=0}^{m-2k+3} \|D_x^l \phi_k(\cdot, x_1) - D_x^l \phi_k(\cdot, x_2)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq \tilde{C}_{m,k,g,\Omega} |x_1 - x_2|.$$

(IP3) $\psi_k \in C^{m-2k+2,1}(\overline{\Omega})$ satisfying

$$\|\psi_k\|_{C^{m-2k+2,1}(\overline{\Omega})} \leq \tilde{C}_{m,k,g,\Omega}.$$

It will turn out at the end that the family $\{w_k : \mathbb{R}^n \times \overline{\Omega} \rightarrow \mathbb{R}\}_{1 \leq k \leq [\frac{m}{2}] + 1}$, defined by

$$(3.3.9) \quad w_k(y, x) = \phi_k(y, x) + \chi^{ij}(y, x) D_{x_i x_j} \psi_{k-2}(x) + \psi_k(x),$$

satisfies Lemma 3.3.2. Here by $\chi^{ij}(y, x)$ we have denoted $D_{p_{ij}} w(y; D_x^2 u, x)$.

Let us make a few remarks on the function $\chi^{ij}(y, x)$, which has the particular importance in what follows. First we observe from Proposition 3.2.1 and Lemma 3.3.1 that $\chi^{ij}(\cdot, x) \in C^{2,\alpha}(\mathbb{R}^n)$ for all $x \in \overline{\Omega}$ and

$$(3.3.10) \quad \|\chi^{ij}(\cdot, x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{m,g,\Omega}^1.$$

In addition, $\chi^{ij}(y, \cdot) \in C^{m-1,1}(\overline{\Omega})$ uniformly for $y \in \mathbb{R}^n$ and

$$(3.3.11) \quad \|\chi^{ij}(y, \cdot)\|_{C^{m-1,1}(\overline{\Omega})} \leq C_{m,g,\Omega}^2,$$

and, in particular for $x_1, x_2 \in \overline{\Omega}$, there holds

$$(3.3.12) \quad \sum_{l=0}^{m-2} \|D_x^l \chi^{ij}(\cdot, x_1) - D_x^l \chi^{ij}(\cdot, x_2)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{m,g,\Omega}^2 |x_1 - x_2|.$$

It is noteworthy to see that, in view of the equation (3.2.9), $\chi^{ij}(\cdot, x)$ solves

$$(3.3.13) \quad a_{rs}(\cdot, x) D_{y_r y_s} \chi^{ij}(\cdot, x) + a_{ij}(\cdot, x) = \bar{a}_{ij}(x) \quad \text{in } \mathbb{R}^n,$$

where $\bar{a}_{ij}(x) = \bar{F}_{p_{ij}}(D_x^2 u, x) \in C^{m-1,1}(\overline{\Omega})$ whose $C^{m-1,1}(\overline{\Omega})$ -norm is bounded above by $C_{m,g,\Omega}^2$.

Let us now begin our induction argument. As the first step, we define $\psi_{-1}(x) = \psi_0(x) = \psi_{[\frac{m}{2}]} = \psi_{[\frac{m}{2}]+1} \equiv 0$ on $\overline{\Omega}$ and $\phi_1(y, x) \equiv 0$, $\phi_2(y, x) = w(y; D_x^2 u, x)$ on $\mathbb{R}^n \times \overline{\Omega}$. If $m = 2$ or 3 , then as we define $\{w_k\}$ according to (3.3.9), we obtain $w_1(y, x) = 0$ and $w_2(y, x) = w(y; D_x^2 u, x)$. The assertions (i) and (ii) of Lemma 3.3.2 are then immediate from Lemma 3.1.2 and Proposition 3.2.1. Since $1 \leq k \leq 2$ when $m = 2$ or 3 , we may neglect the assertion (iii). Thus, Lemma 3.3.2 is proved for the case $m = 2$ and 3 .

Now we consider the case when $m \geq 4$. We see that ϕ_1 and ϕ_2 [resp., ψ_{-1} and ψ_0] chosen in the first step verify (IP1)-(IP2) [resp., (IP3)] by the same reason as above.

In order to run the induction argument, we choose $3 \leq k \leq [\frac{m}{2}] + 1$ and suppose that we have already found the families $\{\psi_{l-2}\}_{1 \leq l \leq k-1}$ and $\{\phi_l\}_{1 \leq l \leq k-1}$ [resp.,

$\{w_l\}_{1 \leq l \leq k-1}$ which satisfy (IP1)-(IP3) [resp., Lemma 3.3.2]. We then define $\tilde{\Phi}_k : \mathbb{R}^n \times \bar{\Omega} \rightarrow \mathbb{R}$ by

$$(3.3.14) \quad \begin{aligned} \tilde{\Phi}_k &= a_{ij} D_{x_i x_j} (\phi_{k-2} + \chi^{ab} D_{x_a x_b} \psi_{k-4}) + 2a_{ij} D_{x_i y_j} (\phi_{k-1} + \chi^{ab} D_{x_a x_b} \psi_{k-3}) \\ &+ \sum_{l=2}^{k-2} \frac{1}{l!} \sum_{n_1 + \dots + n_l = k-2} a_{i_1 j_1 \dots i_l j_l} X_{i_1 j_1}^{n_1} \cdots X_{i_l j_l}^{n_l}. \end{aligned}$$

One may notice that $\tilde{\Phi}_k$ does not involve the functions ψ_{k-2} and ϕ_k neither ψ_{r-2} and ϕ_r for $r \geq k$.

Consider the following problem: For each $x \in \bar{\Omega}$, there exists a unique constant $\Psi_{k-2}(x)$ such that the following PDE,

$$(3.3.15) \quad a_{ij}(\cdot, x) D_{y_i y_j} v + \tilde{\Phi}_k(\cdot, x) = \Psi_{k-2}(x) \quad \text{in } \mathbb{R}^n,$$

attains a bounded 1-periodic solution v . Note that $a_{ij}(\cdot, x)$ is uniformly elliptic with the ellipticity constants λ and Λ . Moreover, $a_{i_1 j_1 \dots i_l j_l}(\cdot, x)$ is 1-periodic and belongs to $C^{m-l,1}(\mathbb{R}^n)$ whose $C^{m-l,1}(\mathbb{R}^n)$ -norm is bounded above by $K_{m,l,g,\Omega}$. This fact together with our induction hypotheses, (IP1)-(IP3) and Lemma 3.3.2 (i) and (ii), yields that $\tilde{\Phi}_k(\cdot, x) \in C^{0,\alpha}(\mathbb{R}^n)$ where its $C^{0,\alpha}(\mathbb{R}^n)$ -norm is bounded above by $\tilde{K}_{m,k,g,\Omega}$. Therefore, Lemma 2.1.2 yields that the PDE (3.3.15) is solvable with a $C^{2,\alpha}(\mathbb{R}^n)$ -solution, and denote it by $\phi_k(\cdot, x)$. In particular, let us choose $\phi_k(\cdot, x)$ such that $\phi_k(0, x) = 0$. Since the domain Ω is bounded, $\phi_k(\cdot, x) \in C^{2,\alpha}(\mathbb{R}^n)$ uniformly for $x \in \bar{\Omega}$ and

$$(3.3.16) \quad \|\phi_k(\cdot, x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{m,k,g,\Omega}.$$

It shows that ϕ_k verifies (IP1).

To know the regularity of ϕ_k in x -variable, we utilize Proposition 3.2.1. We know that $a_{i_1 j_1 \dots i_m j_m}(y, \cdot) \in C^{m-l,1}(\bar{\Omega})$ and its $C^{m-l,1}(\bar{\Omega})$ -norm is bounded above by $L_{m,k,g,\Omega}$. Then again by using our induction hypotheses, we obtain $\tilde{\Phi}_k(y, \cdot) \in C^{m-2k+4,1}(\bar{\Omega})$ whose $C^{m-2k+4,1}(\bar{\Omega})$ -norm is bounded above by $\tilde{L}_{m,k,g,\Omega}$. Thus, Proposition 3.2.1 implies that Ψ_{k-2} and $\phi_k(y, \cdot)$ are both $C^{m-2k+4,1}(\bar{\Omega})$ and

$$(3.3.17) \quad \|\Psi_{k-2}\|_{C^{m-2k+4,1}(\bar{\Omega})} + \|\phi_k(y, \cdot)\|_{C^{m-2k+4,1}(\bar{\Omega})} \leq \tilde{C}_{m,k,g,\Omega};$$

in particular, we obtain for any $x_1, x_2 \in \bar{\Omega}$ that

$$(3.3.18) \quad \sum_{i=0}^{m-2k+3} \|D_x^i \phi_k(\cdot, x_1) - D_x^i \phi_k(\cdot, x_2)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq \tilde{C}_{m,k,g,\Omega} |x_1 - x_2|.$$

Hence, ϕ_k satisfies (IP2) as well.

To this end, we choose the function $\psi_{k-2} : \bar{\Omega} \rightarrow \mathbb{R}$ by the solution of

$$(3.3.19) \quad \begin{cases} \bar{a}_{ij} D_{x_i x_j} \psi_{k-2} = -\Psi_{k-2} & \text{in } \Omega, \\ \psi_{k-2} = 0 & \text{on } \partial\Omega. \end{cases}$$

Recall from Lemma 3.1.8 that \bar{a}_{ij} is uniformly elliptic in $\bar{\Omega}$ with the ellipticity constants λ and Λ . Also Proposition 3.2.1 implies that $\bar{a}_{ij} \in C^{m-1,1}(\bar{\Omega})$ whose $C^{m-1,1}(\bar{\Omega})$ -norm is bounded above by $C_{m,g,\Omega}$. Since $\Psi_{k-2} \in C^{m-2k+4,1}(\bar{\Omega})$ with the estimate (3.3.17), there exists a unique solution $\psi_{k-2} \in C^{m-2k+6,1}(\bar{\Omega})$ of (3.3.19) and

$$(3.3.20) \quad \|\psi_{k-2}\|_{C^{m,1}(\bar{\Omega})} \leq C_{\|\bar{a}_{ij}\|_{C^{m-1,1}(\bar{\Omega})}, \Omega} (\|\psi\|_{L^\infty(\Omega)} + \|\Psi\|_{C^{m-2,1}(\bar{\Omega})}) \leq \bar{C}_{m,k-2,g,\Omega}.$$

Thus, ψ_{k-2} satisfies the induction hypothesis (IP3).

For this moment we put $\psi_k = 0$ and define $w_k : \mathbb{R}^n \times \overline{\Omega} \rightarrow \mathbb{R}$ by (3.3.9). (ψ_k will be recovered to a non-trivial function in the $(k+2)$ -th step, which does not affect our final conclusion.) It is then clear from (3.3.10) and (3.3.16) that $w_k(\cdot, x) \in C^{2,\alpha}(\mathbb{R}^n)$ and

$$\|w_k(\cdot, x)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C_{m,k,g,\Omega} + C_{m,g,\Omega}^1 \bar{C}_{m,k-2,g,\Omega} + \bar{C}_{m,k,g,\Omega} = A_{m,k,g,\Omega}.$$

It verifies Lemma 3.3.2 (i). Also (3.3.17), (3.3.11) and (3.3.20) yields that $w_k(y, \cdot) \in C^{m-2k+2,1}(\overline{\Omega})$ and

$$\|w_k(y, \cdot)\|_{C^{m-2k+2,1}(\overline{\Omega})} \leq \bar{C}_{m,k,g,\Omega} + C_{m,g,\Omega}^2 \bar{C}_{m,k-2,g,\Omega} + \bar{C}_{m,k,g,\Omega} = \bar{A}_{m,k,g,\Omega}.$$

In particular, for any pair of $x_1, x_2 \in \overline{\Omega}$, we obtain from (3.3.12), (3.3.18) and (3.3.20) that

$$\sum_{i=0}^{m-2k+1} \|D_x^i w_k(\cdot, x_1) - D_x^i w_k(\cdot, x_2)\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq \bar{A}_{m,k,g,\Omega} |x_1 - x_2|.$$

This verifies Lemma 3.3.2 (ii). Here we make a remark that in the above inequalities. We have considered the term $\bar{C}_{m,k,g,\Omega}$, although it is not necessary for this moment as we have put $\psi_k = 0$. Nevertheless, we have specified it because ψ_k will be determined in the $(k+2)$ -th step; it will be determined to be the constant satisfying (3.3.20) for ψ_k (instead of ψ_{k-2}) at the $(k+2)$ -th step.

As the final step, we combine (3.3.15) and (3.3.19) and obtain that

$$\begin{aligned} & a_{ij}(\cdot, x) D_{y_i y_j} w_k(\cdot, x) + \Phi_k(\cdot, x) \\ &= a_{ij}(\cdot, x) D_{y_i y_j} \phi_k(\cdot, x) + \bar{\Phi}_k(\cdot, x) + [a_{rs}(\cdot, x) D_{y_r y_s} \chi^{ij}(\cdot, x) + a_{ij}(\cdot, x)] D_{x_i x_j} \psi_{k-2}(x) \\ &= \Psi_{k-2}(x) + \bar{A}_{ij} D_{x_i x_j} \psi_{k-2}(x) \\ &= 0 \quad \text{in } \mathbb{R}^n, \end{aligned}$$

which verifies the assertion (iii). The proof now finishes by the induction principle. \square

Remark. As we note in the remark below Proposition 3.2.1, we see how the coupling effect contribute to the regularity of $x \mapsto w_k(y, x)$. If the x and y -variables were decoupled, we would have obtained $w_k(\cdot, x) \in C^{m-k+2,1}(\overline{\Omega})$ instead of $C^{m-2k+2,1}(\overline{\Omega})$.

To this end we define the k -th order interior corrector w_k^ε of (1.1.1) for each $1 \leq k \leq \lfloor \frac{m}{2} \rfloor + 1$ and $\varepsilon > 0$ by

$$w_k^\varepsilon(x) = w_k\left(\frac{x}{\varepsilon}, x\right) \quad (x \in \overline{\Omega}),$$

where w_k 's are given in accordance with Lemma 3.3.2, and define $\eta_m^\varepsilon : \overline{\Omega} \rightarrow \mathbb{R}$ by

$$(3.3.21) \quad \eta_m^\varepsilon = u + \varepsilon w_1^\varepsilon + \cdots + \varepsilon^{\lfloor \frac{m}{2} \rfloor + 1} w_{\lfloor \frac{m}{2} \rfloor + 1}^\varepsilon.$$

Now we are in position to introduce the boundary layer corrector. The underlying idea of seeking the boundary layer corrector is the same as in the linear case; we correct the boundary oscillation occurred by the interior correctors by solving the corresponding boundary value problem (c.f. (2.2.12)). Due to the nonlinearity of the problem (F_ε), however, we cannot find the boundary layer corrector in an order-wise manner. Instead, we consider a boundary value problem which

involves the entire boundary oscillation caused by the interior correctors; i.e., we solve for each $\varepsilon > 0$ the following PDE,

$$(3.3.22) \quad \begin{cases} F(D^2\eta_m^\varepsilon + D^2\theta_m^\varepsilon, x, \varepsilon^{-1}x) = F(D^2\eta_m^\varepsilon, x, \varepsilon^{-1}x) & \text{in } \Omega, \\ \theta_m^\varepsilon = -\eta_m^\varepsilon + g & \text{on } \partial\Omega. \end{cases}$$

One may notice from Lemma 3.3.2 that $\eta_m^\varepsilon \in C^2(\overline{\Omega})$ that the right hand side of (3.3.22) is a uniformly continuous function on $\overline{\Omega}$ for each $\varepsilon > 0$. Thus, Perron's method (Theorem B.0.1) ensures the unique existence of a viscosity solution $\theta_m^\varepsilon \in C(\overline{\Omega})$ of (3.3.22).

3.4. Proof of Main Theorem II. We shall now prove Main Theorem II.

Proof of Theorem 1.2.1. Suppose that $m \geq 4$. The first part of the proof verifies the discussion we made in the beginning of the previous subsection. Fix $\varepsilon_* \in (0, 1)$ and pick any $\varepsilon > 0$. We will skip the calculation if it has already been done in the previous subsection.

In what follows let us denote by r_m the positive integer $[\frac{m}{2}] + 1$. Recall from [resp.,] X^k , $1 \leq k \leq r_m$ [resp., Y^{r_m}]. By Lemma 3.3.2, we have a uniform bound on the matrix norm of X^k , which is independent of ε , namely,

$$(3.4.1) \quad \|X^k(\cdot/\varepsilon, \cdot)\|_{L^\infty(\Omega)} \leq C_{m,k,g,\Omega}.$$

It is then immediately follows that

$$(3.4.2) \quad \sup_{0 < \varepsilon \leq \varepsilon_*} \|Y^{r_m}(\cdot/\varepsilon, \cdot)\|_{L^\infty(\overline{\Omega})} \leq (1 - \varepsilon_*)L_* \frac{1 - \varepsilon_*^{r_m}}{1 - \varepsilon_*} < L_*$$

where $L_* = (1 - \varepsilon_*)^{-1} \max\{1, C_{m,1,g,\Omega}, \dots, C_{m,r_m,g,\Omega}\}$.

In the rest of this proof, we set $\varepsilon \in (0, \varepsilon_*]$ to be fixed. We choose any $x \in \Omega$ and adopt the Taylor expansion of $F(D^2\eta_m^\varepsilon, x, x/\varepsilon)$ with respect to the M -variable up to $(r_m - 1)$ -th order. For brevity, we omit the dependency on $(\varepsilon^{-1}x, x)$. Then, by the choice of our interior correctors w_k^ε , we end up with

$$(3.4.3) \quad \begin{aligned} F(D^2\eta_m^\varepsilon) &= F(X^0 + \varepsilon Y^{r_m}) \\ &= F(X^0) + \sum_{k=1}^{r_m-1} \frac{\varepsilon^k}{k!} F_{p_{i_1 j_1} \dots p_{i_k j_k}}(X^0) Y_{i_1 j_1}^{r_m} \dots Y_{i_k j_k}^{r_m} + R_m^\varepsilon \\ &= F(X^0) + \sum_{k=1}^{r_m-1} \varepsilon^k \sum_{l=1}^k \frac{1}{l!} \sum_{n_1 + \dots + n_l = k} F_{p_{i_1 j_1} \dots p_{i_l j_l}}(X^0) X_{i_1 j_1}^{n_1} \dots X_{i_l j_l}^{n_l} + \tilde{R}_m^\varepsilon \\ &= \tilde{R}_m^\varepsilon, \end{aligned}$$

where

$$\begin{aligned} R_m^\varepsilon &= \frac{\varepsilon_0^{r_m}}{r_m!} F_{p_{i_1 j_1} \dots p_{i_{r_m} j_{r_m}}}(X^0) Y_{i_1 j_1}^{r_m} \dots Y_{i_{r_m} j_{r_m}}^{r_m} \quad \text{for some } \varepsilon_0 \in [0, \varepsilon], \\ \tilde{R}_m^\varepsilon &= R_m^\varepsilon + \sum_{k=1}^{r_m-2} \sum_{r_m-1 \leq n_1 + \dots + n_k \leq r_m k} \frac{\varepsilon^{n_1 + \dots + n_k}}{k!} F_{p_{i_1 j_1} \dots p_{i_k j_k}}(X^0) X_{i_1 j_1}^{n_1} \dots X_{i_k j_k}^{n_k}. \end{aligned}$$

One should note that $F_{p_{i_1 j_1} \dots p_{i_k j_k}}(X^0)$ are exactly the coefficients $a_{i_1 j_1 \dots i_k j_k}$ appearing in (3.3.8). Now due to (3.4.1) and (3.4.2), there hold

$$|R_m^\varepsilon| \leq \tilde{C}_{m,g,\Omega} L_*^{r_m} \varepsilon^{r_m},$$

and thus,

$$|\tilde{R}_m^\varepsilon| \leq |R_m^\varepsilon| + \tilde{C}_{m,g,\Omega} L_*^{(r_m-2)r_m} \varepsilon^{r_m-1} \leq C_0 \varepsilon^{r_m-1}.$$

The second part of this proof is devoted to the establishment of the estimate (1.2.1). The essence is to construct barriers and argue by the comparison principle. Choose $R > 0$ in such a way that $\overline{\Omega} \subset B_R(0)$. Consider the functions $\eta_m^{\varepsilon,\pm} : \overline{\Omega} \rightarrow \mathbb{R}$ defined by

$$(3.4.4) \quad \eta_m^{\varepsilon,\pm} = \eta_m^\varepsilon + \theta_m^\varepsilon \pm (2\lambda)^{-1} C_0 \varepsilon^{r_m-1} (R^2 - |x|^2) \quad (x \in \overline{\Omega}).$$

By the uniform ellipticity of F (structure condition (F2)) and the choice of the boundary layer corrector (3.3.22), there holds

$$F(D^2 \eta_m^{\varepsilon,+}) \leq F(D^2 \eta_m^\varepsilon + D^2 \theta_m^\varepsilon) - C_0 \varepsilon^{r_m-1} = F(D^2 \eta_m^\varepsilon) - C_0 \varepsilon^{r_m-1} \leq 0$$

in the viscosity sense, and

$$\eta_m^{\varepsilon,+}|_{\partial\Omega} \geq \eta_m^\varepsilon + \theta_m^\varepsilon = g.$$

Thus, $\eta_m^{\varepsilon,+}$ is a viscosity supersolution of (F_ε) . In a similar manner, one can verify that $\eta_m^{\varepsilon,-}$ is a viscosity subsolution of (F_ε) . Thus, the comparison principle yields

$$\eta_m^{\varepsilon,-} \leq u^\varepsilon \leq \eta_m^{\varepsilon,+} \quad \text{in } \overline{\Omega}.$$

It then follows that

$$\|u^\varepsilon - \eta_m^\varepsilon - \theta_m^\varepsilon\|_{L^\infty(\Omega)} \leq (2\lambda)^{-1} C_0 \varepsilon^{r_m-1},$$

which proves (1.2.1).

The proof for the case $m = 2$ or 3 shares the same idea presented above, but is simpler. In this case, $\eta_m^\varepsilon(x) = u(x) + \varepsilon^2 w_2(\varepsilon^{-1}x, x)$, and thus, we do not need the expansion (3.4.3); instead we can directly argue as in the second part. The rest of the proof is exactly the same, so is omitted. \square

APPENDIX A. EXISTENCE AND REGULARITY THEORY OF LINEAR EQUATIONS

Let Ω be a bounded domain of \mathbb{R}^n and λ, Λ be constants with $0 < \lambda \leq \Lambda$. We consider

$$(P) \quad \begin{cases} Lu \equiv a_{ij} D_{ij} u + b_i D_i u + cu = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

for some $f \in C(\overline{\Omega})$ and $g \in C(\overline{\Omega})$, where (a_{ij}) is assumed to be uniformly elliptic with constants λ and Λ in the sense of (L2).

We list the essential results in the existence and regularity theory with regard to (P). For details one may refer to [GT].

Proposition A.0.1. (a) (A priori estimates) If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solves (P) and $c \leq 0$, then

$$\|u\|_{L^\infty(\Omega)} \leq C \left(\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\overline{\Omega})} \right),$$

where C depends only on λ, Λ and $\text{diam}(\Omega)$.

- (b) (Schauder estimates) If $u \in C^{k+2,\alpha}(\Omega)$ [resp. $C^{k+2,\alpha}(\overline{\Omega})$] solves (P), $a_{ij}, b_i, c, f \in C^{k,\alpha}(\Omega)$ [resp. $a_{ij}, b_i, c, f \in C^{k,\alpha}(\overline{\Omega})$, $\partial\Omega \in C^{k+2,\alpha}$ and $g \in C^{k+2,\alpha}(\overline{\Omega})$], then

$$\|u\|_{C^{k+2,\alpha}(\Omega)}^* \leq C \left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^{k,\alpha}(\Omega)}^{(2)} \right),$$

$$[\text{resp. } \|u\|_{C^{k+2,\alpha}(\overline{\Omega})} \leq C \left(\|u\|_{L^\infty(\Omega)} + \|g\|_{C^{k+2,\alpha}(\overline{\Omega})} + \|f\|_{C^{k,\alpha}(\overline{\Omega})} \right)]$$

where C depends only on $k, n, \alpha, \lambda, \Lambda, \|a_{ij}\|_{C^{k,\alpha}(\Omega)}, \|b_i\|_{C^{k,\alpha}(\Omega)}^{(1)}, \|c\|_{C^{k,\alpha}(\Omega)}^{(2)}$ [resp. $\|a_{ij}\|_{C^{k,\alpha}(\overline{\Omega})}, \|b_i\|_{C^{k,\alpha}(\overline{\Omega})}, \|c\|_{C^{k,\alpha}(\overline{\Omega})}$ and Ω].

- (c) (Harnack inequality) If $u \in C^2(B_2) \cap C(\overline{B_2})$ with $u \geq 0$ in B_2 satisfy (P), where $f \in C(\overline{B_2})$, then

$$(A.0.5) \quad \sup_{B_1} u \leq C \left(\inf_{B_1} u + \left\| \frac{f}{\lambda} \right\|_{L^\infty(B_2)} \right)$$

where C is a positive constant depending only on $n, \lambda, \Lambda, |b_i|$ and $|c|$.

- (d) (Liouville theorem) If $u \in C^2(\mathbb{R}^n)$ satisfy the equation $Lu = 0$ in \mathbb{R}^n with $b = c = 0$, then either u is a constant or u is unbounded in \mathbb{R}^n .
- (e) ($C^{k+2,\alpha}$ -regularity) If $u \in C^2(\Omega)$ solves (P) (where Ω is possibly unbounded) and $a_{ij}, b_i, c, f \in C^{k,\alpha}(\Omega)$, then $u \in C^{k+2,\alpha}(\Omega)$. On the other hand, if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies (P) and $a_{ij}, b_i, c, f \in C^{k,\alpha}(\overline{\Omega})$, $\partial\Omega \in C^{k+2,\alpha}$ and $g \in C^{k+2,\alpha}(\overline{\Omega})$, then $u \in C^{k+2,\alpha}(\overline{\Omega})$.
- (f) (Existence) If $a_{ij}, b_i, c, f \in C^\alpha(\overline{\Omega})$, $\partial\Omega \in C^{2,\alpha}$ and $g \in C^{2,\alpha}(\overline{\Omega})$, then (P) has a unique $C^{2,\alpha}(\overline{\Omega})$ -solution.

APPENDIX B. EXISTENCE AND REGULARITY THEORY OF NONLINEAR EQUATIONS

Set Ω to be a bounded domain of \mathbb{R}^n and $(M, p, r, x) \mapsto F(M, p, r, x) \in C(\mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \overline{\Omega})$ to be non-increasing in r . Consider the equation

$$(NP) \quad \begin{cases} F(D^2u, Du, u, x) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

The notion of viscosity solutions for (NP) can be found in many literature; e.g., see [CIL] or [CC]. Here we summarize the existence theory of (NP). The precise statement of the structure conditions (C1)-(C4) appearing in below can be found in [CIL] and [T]. Note that $USC(\Omega)$ and $LSC(\Omega)$ respectively denote the space of all upper and lower semicontinuous continuous functions on Ω .

Theorem B.0.1. (a) (Comparison principle) Suppose $F \in C(\mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega)$ to be (C1) degenerate [resp., uniformly] elliptic (C2) strictly decreasing [resp., non-increasing] in r and (C3) to have a modulus of continuity with respect to p, r and x . Then (NP) has a comparison principle.

Moreover, if $F \in C(\mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$ is 1-periodic in y and satisfies (C1), (C2) (only strictly decreasing) and (C3), then (NP) has a comparison principle for 1-periodic sub- and super-solutions.

- (b) (Perron's method) Suppose that (NP) has a comparison principle. Assume that u_- is a viscosity subsolution of (NP) and u_+ is a viscosity supersolution of (NP). Then there exists a unique viscosity solution u of (NP) satisfying $u_- \leq u \leq u_+$ in $\overline{\Omega}$.

Besides, assume that (NP) is defined on a periodic domain and has a comparison principle. If there are 1-periodic viscosity sub- and super-solutions, then there exists a unique 1-periodic viscosity solution of (NP).

Since the proofs of the second part of both (a) and (b) are not contained in the references which we mentioned above, we present them here.

Proof for the periodic case. We deal only with the case when F depends on the Hessian and the periodic variable and with C^2 solutions; it contains the basic idea of this proof. The general argument then can be made analogous to the proof of the first part of (a) and (b), which is contained in [CIL].

Suppose that u_- [resp., u_+] be a bounded 1-periodic subsolution [resp., supersolution] of $F(D^2u, y) = 0$ in \mathbb{R}^n and that $u_{\pm} \in C^2(\mathbb{R}^n)$. Assume toward a contradiction that $u_-(y_0) > u_+(y_0)$ for some $y_0 \in \mathbb{R}^n$. Then we subtract a positive constant t from u_- so that $u_- \leq u_+$ in \mathbb{R}^n . This is assured by the boundedness of u_{\pm} . Now we slide the solution u_- upward until it touches u_+ by below; to be precise, we choose

$$t_* = \inf\{t \in \mathbb{R} : u_- - t \leq u_+ \text{ in } \mathbb{R}^n\},$$

which is positive because of the assumption that $u_-(y_0) > u_+(y_0)$. Then $u_- - t_*$ touches u_+ by below at a point $y_1 \in \mathbb{R}^n$; i.e.,

$$u_-(y) - t_* \leq u_+(y) \text{ for any } y \in \mathbb{R}^n \quad \text{and} \quad u_-(y_1) - t_* = u_+(y_1).$$

In other words, the function $w(y) = u_+(y) - (u_-(y) - t_*)$ attains its local minimum at y_1 , and since y_1 is an interior point, we must have $D^2w(y_1) > 0$. Then the uniform ellipticity of F yields that

$$0 \geq F(D^2u_+(y_1), y_1) \geq F(D^2(u_-(y_1) - t_*), y_1) + \lambda \|D^2w(y_1)\| > 0,$$

which is a contradiction. Thus, we conclude that $u_- \leq u_+$ in \mathbb{R}^n and complete the proof of (a).

For the Perron's method for periodic solutions, we consider

$$u(y) = \sup\{v(y) : v \text{ is bounded 1-periodic and solves } F(D^2v, y) \geq 0 \text{ in } \mathbb{R}^n\},$$

where the underlying set of the supremum is nonempty due to the assumption. The rest of the proof follows exactly to the argument in [CIL], so we omit the details. \square

For the rest of this section, we focus ourselves to

$$(NP') \quad F(D^2u, x) = f(x) \quad \text{in } \Omega,$$

where $f \in C(\overline{\Omega})$. For details, see [CC].

Theorem B.0.2. (a) (Harnack inequality) Suppose $u \in S^*(\lambda, \Lambda, f)$ and $u \geq 0$ in Q_1 . Then,

$$(B.0.6) \quad \sup_{Q_{1/2}} u \leq C \left(\inf_{Q_{1/2}} u + \|f\|_{L^n(Q_1)} \right)$$

where $C > 0$ depends only on n, λ and Λ .

(b) (Interior C^α -regularity) If $u \in S^*(\lambda, \Lambda, f)$ in Q_1 , then $u \in C^\alpha(\overline{Q}_{1/2})$ and

$$\|u\|_{C^\alpha(\overline{Q}_{1/2})} \leq C(\|u\|_{L^\infty(Q_1)} + \|f\|_{L^\infty(Q_1)})$$

where $0 < \alpha < 1$ and $C > 0$ depend only on n, λ and Λ .

- (c) (*Liouville theorem*) Any bounded below (or above) function which belongs to $S(\lambda, \Lambda, 0)$ in \mathbb{R}^n is constant.
- (d) (*Modulus of continuity*) Let Ω satisfy the uniform exterior sphere condition (with the radius R). Suppose $u \in S(\lambda, \Lambda, f)$ in Ω . Let $\varphi := u|_{\partial\Omega}$ and let $\rho(|x - y|)$ be a modulus of continuity of φ . Then there exists a modulus of continuity ρ^* of u in $\overline{\Omega}$, which depends only on $n, \lambda, \Lambda, \text{diam}(\Omega), R, \|\varphi\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(\Omega)}$ and ρ .
- (e) (*Interior $C^{2,\alpha}$ -regularity*) Suppose that F is concave with respect to M . Moreover, let $x_0 \in \Omega$ and assume that for some $0 < \alpha < 1$ and $r_0 > 0$ with $B_{r_0}(x_0) \subset \Omega$, there hold $\beta(\cdot, x_0), f \in C^\alpha(B_{r_0}(x_0))$, and that u is a viscosity solution of (NP') in $B_{r_0}(x_0)$. Then $u \in C^{2,\alpha}(\overline{B}_{r_1}(x_0))$ for $r_1 = C^{-1}r_0$ and

$$\|u\|_{C^{2,\alpha}(\overline{B}_{r_1}(x_0))}^* \leq C \left(\|u\|_{L^\infty(B_{r_0}(0))} + r_0^2 (\|f\|_{C^\alpha(B_{r_0}(x_0))} + 1) \right)$$

where $C > 1$ depends only on $n, \lambda, \Lambda, \alpha$ and $\|\beta(\cdot, x_0)\|_{C^\alpha(B_{r_0}(x_0))}$.

- (f) (*$C^{k+2,\alpha}$ -regularity*) Suppose that $F \in C^{m,1}(\mathcal{S}^n \times \overline{\Omega})$ and $f \in C^{m,1}(\overline{\Omega})$. If $u \in C^{2,\alpha}(\Omega)$ [resp. $u \in C^{2,\alpha}(\overline{\Omega})$] is a solution of (NP) [resp. (NP) , $\partial\Omega \in C^{m+2,1}$ and $u|_{\partial\Omega} \in C^{m+2,1}$], then $u \in C^{m+2,\alpha}(\Omega)$ [resp. $u \in C^{m+2,\alpha}(\overline{\Omega})$].

APPENDIX C. COMPACT EMBEDDING

We state the compact embedding result of Hölder spaces on the n -dimensional torus. In general, one may replace the torus by a compact metric space. For a proof, see [GT].

Theorem C.0.3. $C^{k,\alpha}(\mathbb{T}^n)$ is embedded compactly into $C^{l,\beta}(\mathbb{T}^n)$ whenever $j + \beta < k + \alpha$ for $k, l \in \mathbb{Z}$ and $\alpha, \beta \in [0, 1]$; moreover, if $\{u_m\}_{m=1}^\infty \subset C^{k,\alpha}(\mathbb{T}^n)$ converges to u with respect to the norm $\|\cdot\|_{C^{k,\beta}(\mathbb{T}^n)}$ where $\beta \in [0, \alpha)$, then $u \in C^{k,\alpha}(\mathbb{T}^n)$ and

$$\|u\|_{C^{k,\alpha}(\mathbb{T}^n)} \leq \sup_{m \geq 1} \|u_m\|_{C^{k,\alpha}(\mathbb{T}^n)}.$$

Acknowledgements Sunghan Kim was supported by NRF(National Research Foundation of Korea) Grant funded by the Korean Government(NRF-2014-Fostering Core Leaders of the Future Basic Science Program). Ki-Ahm Lee was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP) (No.2014R1A2A2A01004618). Ki-Ahm Lee also hold a joint appointment with the Research Institute of Mathematics of Seoul National University.

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